Pathwise Differentiability for SDEs in a convex Polyhedron with oblique Reflection

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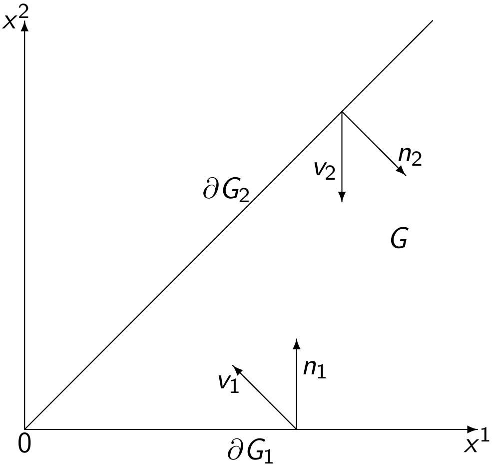
First Example: The two-dimensional Wedge



Properties of the process

- Domain $G \subseteq \mathbb{R}^d$ such that

$$G = \bigcap_{i=1}^{N} G_i, \qquad G_i := \{x : \langle x, n_i \rangle \geq c_i\},$$

  with direction of reflection v_i at ∂G_i.
- We consider a Markov process $(X_t)_{t\geq 0}$ in a convex polyhedron with continuous sample paths such that:
  - In the interior of the polyhedron the process behaves like the solution of an SDE.
  - At the boundary ∂G_i it reflects instantaneously in direction of the constant vector v_i.
  - The process is stopped, when it hits two faces of the polyhedron simultaneously.

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Skorohod-SDE (Oblique Reflection)

For $x \in G$:

$$X_t(x) = x + \int_0^t b(X_r(x))\,dr + w_t + \sum_{i=1}^N v_i\, l_t^i(x), \quad t \geq 0,$$

$$X_t(x) \in G, \quad dl_t^i(x) \geq 0, \quad \int_0^\infty \mathbb{1}_{G \setminus \partial G_i}(X_t(x))\,dl_t^i(x) = 0,$$

where

- $b \in C^1(G)$ and Lipschitz continuous,
- $(w_t)_{t \geq 0}$ d-dimensional Brownian motion.

Problem and Notation

Is the mapping $x \mapsto X_t(x)$, $0 \leq t < \tau$, pathwise differentiable and how do the derivatives (η_t) evolve, where

$$\tau := \inf\{t \geq 0 : X_t(x) \in \partial G_i \cap \partial G_j;\ i \neq j\}?$$

Set

$$C^i := \{t < \tau : X_t(x) \in \partial G^i\}, \qquad r_i(t) := \sup([0,t] \cap C^i)$$

and

$$s(t) := \begin{cases} 0 & \text{no hit of } \partial G \text{ before } t, \\ i & \text{last hit of } \partial G \text{ before } t \text{ in } \partial G_i. \end{cases}$$

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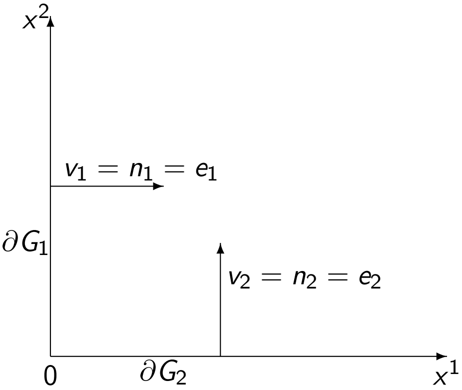
The Case $G = \mathbb{R}^d_+$ and $v_i = n_i$



Results of Deuschel/Zambotti (2005)

Theorem

For all $t \geq 0$ and all $x \in \mathbb{R}^d_+$, a.s. the map $x \mapsto X_t(x)$, $t \geq 0$, is continuously differentiable and, setting $\eta^{ij}_t := \frac{\partial X^i_t(x)}{\partial x^j}$, $i, j \in \{1, \dots, d\}$, there exists a right continuous modification of η such that we have a.s. for all $t \geq 0$:

$$\eta^{\cdot j}_t = \delta_{\cdot j} + \sum_{k=1}^{d} \int_0^t \frac{\partial b}{\partial x^k}(X_r(x))\, \eta^{kj}_r \, dr, \qquad s(t) = 0,$$

$$\eta^{\cdot j}_t = \begin{pmatrix} \vdots \\ \eta^{(i-1)j}_{r_i(t)} \\ 0 \\ \eta^{(i+1)j}_{r_i(t)} \\ \vdots \end{pmatrix} + \sum_{k=1}^{d} \int_{r_i(t)}^t \frac{\partial b}{\partial x^k}(X_r(x))\, \eta^{kj}_r \, dr, \qquad s(t) = i.$$

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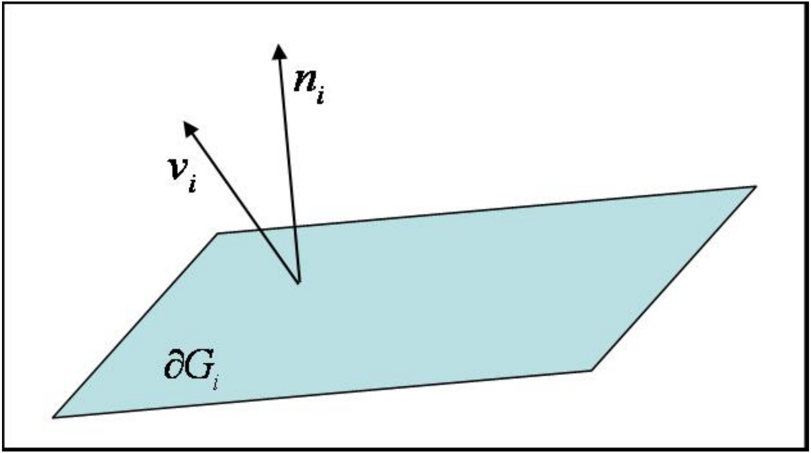
$$\eta_t^{\cdot j} = \sum_{k \neq i} \langle \eta_{r_i(t)}^{\cdot j}, e_k \rangle e_k + \sum_{k=1}^{d} \int_{r_i(t)}^t \frac{\partial b}{\partial x^k}(X_r(x))\, \eta_r^{kj}\, dr, \quad s(t) = i.$$

Choice of the Coordinate System (1)



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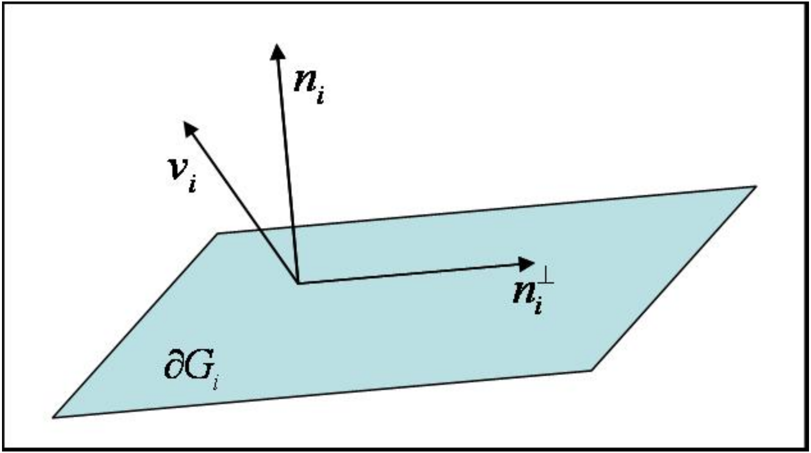

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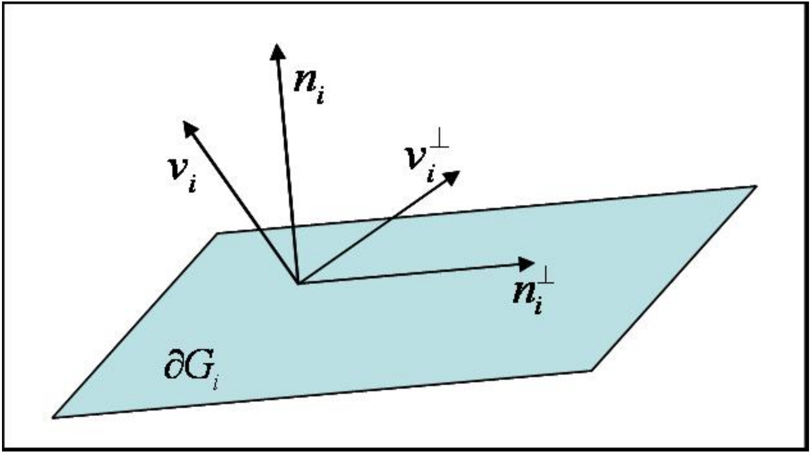

Choice of the Coordinate System (2)

- For every ∂G_i we are given n_i and v_i normalized such that $\langle v_i, n_i \rangle = 1$.
- We choose $v_i^\perp, n_i^\perp \in \text{span}\{n_i, v_i\}$ such that

$$\langle v_i, v_i^\perp \rangle = \langle n_i, n_i^\perp \rangle = 0, \quad \langle n_i^\perp, v_i^\perp \rangle = \langle n_i, v_i \rangle = 1, \quad \langle n_i, v_i^\perp \rangle > 0,$$

- Let $(n_i^k)_{k=3,\dots,d}$ be such that it completes $\{n_i, n_i^\perp\}$ to an orthonormal basis in $\mathbb{R}^d$.

Result

Theorem

The mapping $x \mapsto X_t(x)$, $x \in G$, is differentiable a.s. for all $t \in [0,\tau)\backslash C$ and, setting $\eta_t^{ij} := \partial X_t^i(x)/\partial x^j$, $i,j \in \{1,\dots,d\}$, there exists a right continuous extension of η on $[0,\tau)$, which has a.s. the following form:

If $s(t) = 0$:

$$\eta_t^{\cdot j} = \delta_{\cdot j} + \int_0^t \sum_{k=1}^d \frac{\partial b}{\partial x^k}(X_r(x))\eta_r^{kj}\, dr,$$

and if $s(t) = i$:

$$\eta_t^{\cdot j} = \langle \eta_{r_i(t)}^{\cdot j}, v_i^\perp \rangle n_i^\perp + \sum_{k=3}^d \langle \eta_{r_i(t)}^{\cdot j}, n_i^k \rangle n_i^k + \int_{r_i(t)}^t \sum_{k=1}^d \frac{\partial b}{\partial x^k}(X_r(x))\eta_r^{kj}\, dr.$$

Observations

Let t_i be a time when the process X reaches ∂G_i. Then:

- $\eta^{\cdot j}$ is projected to the tangent space at time t_i:

$$\langle \eta^{\cdot j}_{t_i}, n_i \rangle = 0.$$

- $\eta^{\cdot j}$ jumps at time t_i as follows:

$$\eta^{\cdot j}_{t_i} - \eta^{\cdot j}_{t_i -} = -\langle \eta^{\cdot j}_{t_i -}, n_i \rangle v_i.$$

- The process $(X_t, \eta^{\cdot j}_t)$ in $G \times \mathbb{R}^d$ is Markovian!!

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Generator L of $(X_t, \eta_t^{\cdot j})$ (1)

$$LF(x,\eta) := L^1 F(.,\eta)(x) + L^2 F(x,.)(\eta),$$

The marginal generator of X:

$$L^1 F(.,\eta)(x) := \frac{1}{2}\Delta F(.,\eta)(x) + \sum_{i=1}^{d} b^i(x)\,\frac{\partial F}{\partial x^i}(.,\eta)(x),$$

where $F(.,\eta) \in C^2(G)$ satisfies the Neumann boundary condition:

$$D_{v_i} F(.,\eta)(x) = 0 \qquad \text{for } x \in \partial G_i,$$

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Generator L of $(X_t, \eta_t^{\cdot j})$ (2)

$$LF(x,\eta) := L^1 F(.,\eta)(x) + L^2 F(x,.)(\eta),$$

The marginal generator of η depending on x:

$$L^2 F(x,.)(\eta) := \sum_{i=1}^{d} \left(\sum_{k=1}^{d} \frac{\partial b^i}{\partial x^k}(x)\, \eta^k \right) \frac{\partial F}{\partial \eta^i}(x,.)(\eta),$$

where $F(x,.) \in C^1(\mathbb{R}^d)$ satisfies the Dirichlet boundary condition

$$F(x,\eta) = F(x, \eta - \langle \eta, n_i \rangle n_i), \qquad x \in \partial G_i$$

and the Neumann condition:

$$D_{v_i} F(x,.)(\eta) = 0, \qquad x \in \partial G_i.$$

Open Problems

- What happens for $t > \tau$?
- Random walk representation for the derivatives?
- What about general domains G with smooth boundary?

References

- J.-D. Deuschel, L. Zambotti, (2005), *Bismut-Elworthy formula and Random Walk representation for SDEs with reflection*, Stochastic Process. Appl. 115, 907-925.
- P. Dupuis, H. Ishii, (1991), *On Lipschitz continuity of the solution mapping to the Skorokhod problem, with applications*, Stochastics 35, 31-62.
- P. Dupuis, H. Ishii, (1993), *SDEs with oblique reflection on nonsmooth domains*, The Annals of Prob. 21, No. 1, 554-580.
- P.L. Lions, A.S. Sznitman, *Stochastic Differential Equations with Reflecting Boundary Conditions*, Comm. Pure Appl. Math. 37 (1984), 511-537.