RESEARCH TOPICS

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My work has been centered for many years on

Regularisation and renormalisation methods in geometry, number theory and physics

with the use of analytic tools such as *pseudodifferential operators and symbols*, geometric tools in *differential geometry* and for which I have borrowed ideas from *noncommutative geometry* and *index theory* in mathematics on the one hand and *quantum field theory* in physics on the other hand.

Let me split the description of my work in two periods, a first period before 2003 during which I concentrated on infinite dimensional geometry thereby getting acquainted with and developping some tools which I then further applied to number theory in more recent work from 2003 to 2007, a period during which I simultaneously explored the structures involved in this ψ**do** approach to infinite dimensional geometry and number theory in a comparative manner.

1 BEFORE 2003

- **Infinite dimensional manifolds** Infinite dimensional manifolds naturally arise as configuration or "path" spaces in quantum field theory (Q.F.T.). There are many ways to approach infinite dimensional manifolds; inspired by stochastic approaches to quantum field theory I had learned from my PhD adviser S. Albeverio, I first adopted a stochastic approach. Investigations on stochastic processes on infinite dimensional manifolds along the lines of work by Y. Daletskii gave rise to joint articles with M. Arnaudon (see e.g. [AP1],[AP2], [AP3]) in an attempt to capture some geometric features of a class of infinite dimensional manifolds that naturally arise in quantum field theory. Though these methods proved to be very efficient, this stochastic approach did not seem to capture all the interesting quantities that arise from Q.F.T.; for example, whereas the partition function is one of the first quantities one computes in quantum field theory, from a probabilistic point of view the partition function is hidden in a normalising factor (which yields the analog of a probiblity measure) and hence irrelevant.
- **Pseudo-differential operators** Just as there are both probabilistic and pseudo-differential type proofs for the index theorem, as well as the stochastic point of

view just mentioned, there is also a pseudo-differential approach to infinite dimensional geometry, which I later turned to together with students (A. Cardona, C. Ducourtioux, J.-P. Magnot) and cowokers (M. Arnaudon, Y. Beolopolskaya, S. Rosenberg) (see e.g. [CDMP], [ABP], [P1], [PR1], [PR2]). The basic idea in this approach to infinite dimensional geometry, which is inspired by work by Bismut and Freed on determinant bundles associated with families of Dirac operators as well as by ideas commonly used by physicists, is to substitute to natural but ill-defined geometric concepts in infinite dimensions, their "regularised" counterpart. The mathematical tools used to do so are *regularised traces of pseudodifferential operators* (ψ**dos**) and their counterparts on the level of symbols, namely regularised integrals of pseudodifferential symbols. This is close in spirit to methods used in quantum field theory when e.g. substituting *regularised determinants* to ordinary determinants to describe partition functions. This approach applies to a class of infinite dimensional manifolds and vector bundles which includes "path spaces" (such as loop groups, see [CDMP]), some of which arise in Q.F.T.

- **Weighted traces** In this infinite dimensional setup, in order to regularise otherwise diverging traces, one typically needs some extra geometric data given by an elliptic operator (or a family of such operators) that we call *weight*. Weights (which are typically generalised Laplacians or generalised Dirac operators) offer here a counterpart for Brownian motion in a stochastic approach and play a role similar to the operator D which arises in Fredholm modules in non commutative geometry as a substitute for the metric. The weight used to regularise the otherwise divergent trace leads to *weighted traces*; these occur in many infinite dimensional geometric situations such as in the curvature on the determinant bundle associated with a family of Dirac operators investigated by Bismut and Freed, in the Ricci curvature on the loop space investigated by Freed, the eta invariant of a Dirac operator can also be seen as a weighted trace....

- **Trace anomalies** Using weighted traces as an Ersatz for ordinary traces gives rise to obstructions when attempting to carry out geometric constructions in finite dimensions to the infinite dimensional set up. Such obstructions lie in discrepancies such as the nonvanishing of the Hochschild coboundary of a weighted trace, the failure for regularised integrals to be translation invariant.... familiar to physicists and some of which had already been investigated by mathematicians such as Melrose and Nistor. These descrepancies which we refer to as *trace anomalies* are of purely infinite dimensional nature since they can be expressed in terms of *noncommutative residues* which vanish on finite rank operators. As such they have some locality feature inherited from that of the noncommutative residue.
  Trace anomalies also relate to *anomalies in Q.F.T.* [CDP]. In the chiral setting for example, as it was pointed out by Bismut and Freed, anomalies can be read off the geometry of the determinant bundle associated to a family of Dirac operators; one can relate the curvature of this determinant bundle to tracial anomalies [CDP].
  Trace anomalies are also responsible for local terms arising in various *index type theorems*, a first trivial illustration being the fact that the index of a Fredholm operator can be written as a coboundary of some regularised trace, another less trivial example being the local term in the Atiyah-Patodi-Singer which can be seen as another type of tracial anomaly [CDP]. This interpretation gives an a priori explanation for both the pure infinite dimensional nature of such terms arising in index theorems and their "locality" in the sense that, being tracial anomalies, they have some local description.

- **Circumventing anomalies** Just as one tries to circumvent anomalies in Q.F.T., one would like to get around the various tracial anomalies that arise from insisting on using finite parts of diverging expressions as a substitute for well defined objects such as traces or determinants. There are various ways around, one of which– again inspired by physisicts– consists in adding counterterms to take care of tracial anomalies. Another approach to the problem is to give up taking finite parts of traces of ψ**dos** and to pick instead traces of leading symbols of ψ**dos** . Both these approaches were adopted in [PR1],[PR2] to define characteristic classes on infinite dimensional rank vector bundles.

All these issues lead to further and more recent developments which I describe below.

2 RECENT RESEARCH TOPICS (SINCE 2003)

My research topics in the last few years have reached out to other areas of mathematics such as **number theory** and I was brought to use algebraic tools such as *Hopf algebras*. This lead me to consider further issues such as

- **Renormalisation** The regularisation methods described above which apply to integrals of pseudo-differential symbols (closely related to traces of pseudo-differential operators) are not sufficient to get a grasp on diverging multiple integrals (arising in Feynamn integrals) and multiple sums (arising in number theory) of tensor products of symbols; renormalisation is required to extract from these diverging expressions the "correct" finite part. One can indeed implement renormalisation methods *à la Connes et Kreimer* to renormalise multiple sums and integrals of pseudodifferential symbols, with applications to multiplezeta functions in number theory (see my joint work with D. Manchon [MP1, MP2] and work in progress [BP]) as well as applications to Feynman type integrals, namely to multiple integrals of symbols with linear constraints which provide a toy model for Feynman integrals [P3].

- **Renormalised Chern-Weil form** With the urge to understand the infinite dimensional counterpart of Chern-Weil forms as well as to get a better understanding of anomalies in geometry and physics, I pursued my investigations along these lines using Chern-Weil forms associated with superconnections. These investigations started in joint work with Steven Rosenberg and developped further since then in some joint work with Simon Scott [PS2], further continuing with a joint publication with Jouko Mickelsson [MP],

- **Anomalies** Discrepancies mentioned above that arise from insisting on viewing the infinite dimensional setup as a generalisation of a finite dimensional setup, i.e. from using regularised traces and detemrinants that extend ordinary traces and determinants, occur in various disguises. With the drive to get a better understanding and an overview of these anomalous phenomena, I further continued my investigations along these lines. *Tracial anomalies* for quantised traces is the topic of [P2], *conformal anomalies* is the subject of joint work with Steven Rosenberg [PR3] the failure for *Stoke's formula* to hold for regularised integrals was investigated in [MMP], the *multiplicative anomalies* for determinants was revisited in [OP] in the light of an extended Campbell-Hausdorff formula for pseudodifferential operators.

- **Uniqueness results** In contrast with the noncommutative residue which is canonical as the unique trace on classical pseudodifferential operators by a celebrated result of Wodzicki, regularised traces which extend the usual trace seem rather uncanonical at first sight. However, it turns out as was shwon by Maniccia, Schrohe and Seiler that the canonical trace on non integer order classical pseudo-differential operators, which is based on an Hadamard finite part regularisation procedure is canonical in as far as it is the unique linear extension of the ordinary trace which vanishes on (non integer order) brackets.

We explored and generalised these characterisations in two ways. In [LP] we proved the uniqueness of multiplicative determinants on the grounds of a characterisation of traces on zero order classical pseudodifferential operators which include the noncommutative residue. In the light of Stokes' property for integrals of symbols, I further related the uniqueness of the canonical trace on non integer order classical pseudodifferential operators with the uniqueness of extensions of the ordinary integral to linear forms on non integer classical symbols with Stokes' property [P4].

Below are listed a few concrete results among others obtained in these papers:

- With Jean-Marie Lescure [LP], we characterised **multiplicative determinants** for elliptic operators on a closed manifold, showing that they are built from two types of determinants, the expected "residue determinant" and a new (to our knowledge) determinant associated with the leading symbol trace introduced previoulsy in joint work of mine with S. Rosenberg.
- With Simon Scott [PS1], we generalised the known relation between the noncommutative (or Wodzicki) residue and the complex residue by providing an explicit formula for the coefficients of the **Laurent expansion of the canonical trace** of a holomorphic family of classical pseudodifferential operators.
- Using the coefficients of the Laurent expansion of the canonical trace of holomorphic families of operators derived in [PS1], together with Steven Rosenberg [PR3], we could then provide a unified description of known **conformal invariants** on closed Riemannian manifolds.
- With Jouko Mickelsson [MP] inspired by work with Simon Scott on **Chern-Weil forms associated with a superconnection** [PS2] as well as by previous work by John Lott, we established a transgression formula for an odd Chern form associated with a superconnection built from the (super) noncommutative residue extended to pseudodifferential valued forms.
- In [P2] I reinterpreted the locality of tracial anomalies as a consequence of locality of *quantised* **regularised traces**, by using techniques borrowed from Connes, Moscovici and Higson.
- With Dominique Manchon, we showed [MP2] the **rationality of renormalised multiplezeta values** at non positive integer points (which statisfy stuffle relations), following the renormalisation approach à la Connes Kreimer which we adapted in that same paper to the multiplezeta function setup.

3 ONGOING RESEARCH PROJECTS

A common symbolic approach In all the work described above, pseudodifferential symbols play a fundamental role, whether arising in sums as in multiplezeta functions, in integrals as in Feynman integrals, or as symbols of operators in traces used to define Chern-Weil forms or in determinants associated with elliptic operators.

Tensor algebras of symbols Whereas regularisation procedures are implemented on functionals on algebras of symbols or operators, renormalisation procedures (whether applied to multiplezeta functions or to multiple integrals with linear constraints) are implemented on functionals on tensor algebras of symbols. Indeed, multiplezeta functions seen as sums of symbols on certain cones, resp. multiple integrals with linear constraints all give rise to functionals on tensor products of symbols; whereas the first are renormalised in such a way as to preserve the stuffle product, the latter are renormalised in order to preserve multiplicativity under tensor products (concatenation of diagrams in the language of Feynman diagrams). The symbolic approach therefore reveals strong analogies between two a priori seemingly different objects, multiplezeta functions on the one hand and integrals with linear constraints on the other hand which provide a toy model for Feynman integrals. Transporting the Connes and Kreimer renormalisation procedure from Feynman integrals to multiple sums of symbols turned out to be fruitful to renormalise multiplezeta functions.

Regularisation versus renormalisation I believe there is yet more to be understood from these analogies. Multiple zeta functions seen here as Chen sums of symbols bare the advantage over multiple integrals of symbols with linear constraints that the conical constraints that underly their definition are easier to handle than linear constraints. In spite of this apparent simplicity, the links between various renormalisation procedures one can implement to define renormalised multiplezeta functions that obey stuffle relations are not fully understood (see below). In particular, it is not clear how renormalisation procedures based on zeta regularisation type methods [MP] on the one hand relate to those based on heat-kernel type methods (see the work of Berline and Vergne, and the work of Guo and Zhang). This raises the important question how renormalisation procedures depend precisely on the underlying regularisation procedure, a question which to my knowledge is not yet settled for renormalised multiplezeta functions which obey stuffle relations. Similar issues arise in perturbative quantum field theory where the renormalisation group takes care of some scaling freedom left once has chosen a regularisation procedure; it is however yet unclear to me how much renormalisation procedures depend precisely on the underlying regularisation method used.

Let me now more concretely describe what I am currently working on.

I am finishing a **monograph** (lecture notes), the aim of which is to present various regularisation methods used in infinite dimensional geometry, number theory and quantum field theory, which strongly rely on properties of pseudodifferential symbols. I was given a sabbatical year by the CNRS ("Délégation CNRS") to carry out this project which is well underway.

Here are some **ongoing research projects** which should contribute to a better understanding of renormalisation, resp. regularisation of sums and integrals of symbols,resp. regularised traces of operators, which I claim lie at the heart of various renormalisation issues as well as issues in infinite dimensional geometry.

- **Alternative renormalisation methods** (work in progress with M. Vergne) Renormalised multiplezeta values at non positive integer points can be seen as renormalised sums of polynomials at integer points of a rational cone in view of recent work of N. Berline and M. Vergne. Their work clearly gives another renormalisation method to define multiplezeta values at non positive integer arguments which satisfy stuffle relations (this corresponds to a valuation property in their language) and raises the question of the relation between the various renormalisation procedures one can implement for that purpose (there are at least five at this stage!).

- **Manifolds with boundary** (work in progress with G. Grubb et R. Nest) We are preently investigating the generalisation to manifolds with boundary of the explicit formula obtained with S. Scott on the coefficients of the Laurent expansion for the canonical trace of a holomorphic family of operators.

- **Renormalisation of iterated integrals of symbols with affine constraints** Feynman integrals can be viewed as interated integrals of symbols with affine constraints. In the absence of exterior momenta, the constraints become linear, a situation investigated in [P3]. I hope to extend the renormalisation procedure described there to affine constraints in order to take into account exterior momenta.

- **Multiplezeta functions associated with elliptic operators** The Riemann zeta function generalises to zeta functions associated to elliptic operators on closed manifold. Similarly, one can associated multiplezeta functions to elliptic operators on a closed Riemannian manifold. I hope to investigate the geometric information the asymptotics of multiplezeta functions associated with Lapalce-Beltrami operators could contain when using a heat-kernel regularisation approach.

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