DIRECT CONNECTIONS ON GROUPOIDS AND THEIR JET PROLONGATIONS.

Abstract. Direct connections are to Lie groupoids, what connections are to principal bundles. They can serve as a substitute for differentiation in a non smooth setup and arise in Hairer's regularity structures, a theory taylored to solve stochastic PDEs. We first review the concept of groupoid and define direct connections. We show that groupoids which admit a direct connection are built from principal bundles and focus on frame groupoids built from frame bundles. From a direct connection on a frame groupoid, on its jet prolongation, we construct two types of direct connections which we discuss and compare. One of these is a projective system of direct connections compatible with the projective structure of jet prolongations. The other one appears in polynomial regularity structures, a toy model in the vast theory of regularity structures, which we discuss from a geometric point of view using direct connections on groupoids.

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This paper is dedicated to the memory of the late Kirill Mackenzie (1951-2020)

Contents

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Introduction

This expository paper is dedicated to the study of groupoids equipped with direct connections and to their jet prolongations. Our original motivation was to get a better geometric understanding of the theory of regularity structures developed by Hairer [H14], which motivated our study of direct connections on jet prolongations of groupoids. We discuss the geometric framework underlying regularity structures at the end of the paper. Beyond the new results it presents, this article provides a pedagogical presentation on groupoids with connections and their jet prolongations, which is accessible to non experts in one or more of these subjects. An abridged version should be available in the near future.

Direct connections arise under various disguises in algebraic, geometric and analytic contexts:

- Direct connections were first introduced by Teleman [Te04, Te07, KT06] in the context of non commutative geometry under the name *linear direct connection*. They were built on frame groupoids in order to define the Chern character of the tangent bundle of a smooth manifold from the geodesic distance function by means of cyclic homology. Quoting Teleman, "while a linear connection provides a transport of fibers along curves, a linear direct connection provides a direct transport of fibres from point to point", so that "direct connections [can] be defined in contexts where differentiability is not available".
- Direct connections arise as *re-expansion (or transport) maps* in Hairer's regularity structures [H14, H2] on an Euclidean vector space and were later generalised to a Riemannian manifold in [DDD19]. Here again they arise in the context of singularities, since regularity structures offer an algebraico-analytic device to transform a singular stochastic differential equation into a fixed point problem. Hairer's approach involves an ad hoc *Taylor expansion* of the solutions at any point in space-time and a collection of *re-expansion maps* which relate the values of Taylor expansions at different points.
- Direct connections on groupoids arise in [Koc89] and compare (modulo an extra symmetry requirement which amounts to trivial torsion) with *1-forms* discussed by Kock [Koc07, Koc17] in the context of synthetic geometry, an approach to differential geometry inspired by ideas of Grothendieck.
- For connections with trivial curvature, we recover *local morphisms from the pair groupoid to a general groupoid* studied by Mackenzie in [MK05], a reference textbook on groupoids on which much of this paper is based.

We furthermore expect direct connections to play a role in higher gauge theory when viewing groupoids equipped with direct connections as a generalisation of principal bundles with connections that are ubiquous in gauge theory.

The study of the jet prolongation of groupoids with connections that we undertake in this paper was prompted by the quest for a consistent geometric framework to host the abstract Taylor expansions that feed into Hairer's approach. The work of Diehl, Driver and Dahlqvist [DDD19] confirms that frame groupoids with direct connections (which the authors call transportation maps) play a central role in a geometric approach to regularity structures.

The paper is organised in five sections with the last one dedicated to direct connections in the context of regularity structures, see Theorem 5.6. It relies on the previous section which discusses direct connections on jet groupoids containing the main new results, namely Theorems 4.5, 4.6 and 4.11. Along the way, we prove intermediate results on direct connections such as Proposition 2.4 and Theorem 2.16 in Section 2, which to our knowledge are new. In Section 3, we mostly review known results on jet prolongations of groupoids, organising them in a systematic presentation which we feel is accessible to the non-expert.

Lie groupoids and algebroids. Section 1 is a review of known results on groupoids with a focus on gauge groupoids.

A *Lie groupoid* $\mathcal{G} \rightrightarrows M$ on a smooth manifold M is a smooth collection of elements, called *arrows*, above pairs of points in a manifold, endowed with a partial associative and unital multiplication compatible with the base points such that all arrows are invertible, all involved maps are smooth and the projections of an arrow to its source and target points in M are surjective submersions. Lie groupoids can then be seen as a (bi-)fibred generalisation of Lie groups which can act on fibre bundles keeping track of both the fibre transformations (internal symmetries) and the bundle automorphisms (global symmetries) [Wei96]. They are therefore well suited to describe extended notions of symmetries in many contexts of mathematics and physics [Br06, Hi71, Lan06].

We shall focus on *gauge groupoids*, also called *Atiyah groupoids*, which are locally trivial Lie groupoids, and those among the Lie groupoids that can be equipped with a direct connection. We recall (Proposition 1.5) the one to one correspondence between gauge groupoids and principal bundles

$$P \longmapsto \mathcal{G}(P) = P \times_G P, \tag{1}$$

which sends a principal bundle $P \to M$ to the corresponding gauge groupoid $\mathcal{G}(P) \rightrightarrows M$. Lie groupoids are a generalisation of Lie groups, and as Lie groups, they are locally determined by their infinitesimal structure, given by Lie algebroids.

To a vector bundle $E \to M$ corresponds a canonical groupoid $\mathrm{Iso}(E) \rightrightarrows M$ whose arrows are all possible isomorphisms between any two fibres, called the *frame groupoid*. The frame groupoid coincides with the gauge groupoid of the canonical *frame bundle* of $E \to M$. The Lie algebroid $T^{\mathrm{lin}}E$ of $\mathrm{Iso}(E)$ consists of vertical vector fields on E which are linear and we discuss the map that sends such a vector field to a linear derivation on E, see eq. (14) in §1.6. This establishes an isomorphism between $\mathcal{L}(\mathrm{Iso}(E))$ and the bundle $\mathrm{Der}(E)$ of linear derivations on E.

In view of the applications we have in mind, we consider reduced frame groupoids obtained as the gauge groupoid of a reduced frame bundle (Proposition 1.14). We end the section with a short discussion on local bisections which later enter the construction of jet prolongations of groupoids. In particular, it is useful to observe that local bisections on a gauge groupoid amount to automorphisms of the underlying principal bundle (Example 1.15).

Direct connections on Lie groupoids. In Section 2, we introduce our main protagonists, direct connections on groupoids, and study their properties. If the base manifold M has an affine connection, such as the Levi-Civita connection on a Riemannian manifold, a linear connection ∇ on the bundle $E \to M$ induces a local parallel transport among fibres, along geodesics of M, that is, a linear isomorphism $\tau(x, y) : E_y \to E_x$ for any pair of points (x, y) of M sufficiently close. The parallel transport τ along geodesics is an instance of a general *direct connection* of the gauge groupoid of E, called *linear direct connection* by N. Teleman in [Te04].

We generalise Teleman's linear direct connections to *direct connections* on a groupoid $\mathcal{G} \rightrightarrows M$, which are local maps $\Gamma : \mathcal{P}(M) \nrightarrow \mathcal{G}$ defined on a neighborhood of the identity in the pair groupoid $\mathcal{P}(M) \rightrightarrows M$ with values in $\mathcal{G}$. Groupoids with direct connection, our main object of study, are locally trivial (Proposition 2.4) and hence gauge groupoids $\mathcal{G}(P)$ built from a principal bundle P.

Differentiating a direct connection $\mathcal{G}(P) \rightrightarrows M$ on a gauge groupoid along the diagonal gives rise to a connection on the underlying principal bundle $P \to M$ (Proposition 2.11). It is given by an *infinitesimal connection* on $\mathcal{G}(P)$, namely a vector bundle morphism $\delta^\Gamma : TM \longrightarrow A(P)$ where TM is the tangent bundle of M and $A(P)$ the Atiyah bundle which corresponds to the Lie algebroid of $\mathcal{G}(P)$ (Proposition 2.11). To construct a direct connection from an infinitesimal connection, one can use a parallel transport along geodesics (Proposition 2.13) built from a connection on the underlying manifold. Theorem 2.16 shows that if Γ is the direct connection defined by a parallel transport on P, then ∇^Γ coincides with the classical connection related to the parallel transport. Yet not every direct connection is of this form (Example 2.18), and there is no bijective correspondence between infinitesimal and direct connections.

The *curvature of a direct connection* $\Gamma : \mathcal{P} \twoheadrightarrow \mathcal{G}$ on a groupoid $\mathcal{G} \rightrightarrows M$ is an obstruction to Γ defining a local morphism, given by $\Omega_\Gamma(x, y, z) = \Gamma(z, x)^{-1}\,\Gamma(z, y)\,\Gamma(y, x)$ defined on triples (x, y, z) of pairwise neighboring points in M (eq. (35) in §2.7). The connection is flat when $\Omega_\Gamma = \mathrm{Id}$ and flatness of connections is preserved by differentiation, as well as by integration. There is a one-to-one correspondence between flat infinitesimal connections and flat direct connections (modulo germ equivalence) on groupoids since flat direct connections are entirely determined by the parallel transport induced by the underlying flat infinitesimal connection (Proposition 2.30).

Jet prolongations of bundles and groupoids. Section 3 is dedicated to prolongations of groupoids first considered by Ehresman [Eh55]. The *n-jet prolongation* of a Lie groupoid $\mathcal{G} \rightrightarrows M$ is the jet space of n-jets of local bisections $\sigma : M \twoheadrightarrow \mathcal{G}$ (eq. (68) in §3.3). It can be equipped with a Lie groupoid structure $J^n\mathcal{G} \rightrightarrows M$ induced by that of $\mathcal{G}$. Later, Kolár [Kol07] showed that the jet prolongation J^n actually defines a functor on gauge groupoids (Proposition 3.3). For a principal bundle $P \to M$, we have $J^n\mathcal{G}(P) = \mathcal{G}(W^nP)$ (eq. (73) in Proposition 3.3). Here, W^nP is the n-jet principal prolongation of P given in eq. (60) in §3.5, which entails both the n-jet prolongation J^nP of the principal bundle P and the n-th frame bundle F^nM of M defined in eq. (46) in §3.2. On the infinitesimal level, eq. (73) yields the corresponding property for Atiyah bundles $J^nA(P) \cong A(W^nP)$ (eq. (84) in §3.9). When $P = FE$ is the frame bundle of a vector bundle $E \to M$, eq. (73) in Proposition 3.3 gives the description of the n-jet prolongation $J^n\mathrm{Iso}(E)$ of the groupoid $\mathrm{Iso}(E) = \mathcal{G}(FE)$ as a gauge groupoid $J^n\mathrm{Iso}(E) \cong \mathcal{G}(W^nFE) = W^nFE \times_{W^n_d GL_r} W^nFE$, see eq. (77). It is is a proper subgroupoid of $\mathrm{Iso}(J^nE)$, see eq. (81) in §3.8.

Direct connections on jet groupoids. In Section 4, we consider direct connections on jet prolongations of Lie groupoids. Proposition 4.2 confirms the fact that a jet prolonged groupoid $J^n\mathcal{G}$ with connection is necessarily a gauge groupoid, namely $J^n\mathcal{G} \cong \mathcal{G}(W^nP)$ with $\mathcal{G} = \mathcal{G}(P)$. From a direct connection Γ on a Lie groupoid $\mathcal{G} \rightrightarrows M$, we build connections on the jet-prolongation $J^n\mathcal{G}$, called *n-th order prolongation of* Γ (see Definition 4.1), whose composition with the jet projection map $\pi^n_0 : J^n\mathcal{G} \to J^0\mathcal{G} = \mathcal{G}$ described in §3.7, gives back Γ.

For this purpose, we assume M comes with an affine connection and first we build a direct connection Δ^M (see eq. (97)) on $\mathrm{Iso}(TM)$ called *exponential direct connection* by means of the *exponential (local) bisection* (Definition 4.3). This uses parallel transport on TM along small geodesics induced by the connection on M.

Taking jets of the exponential bisection gives rise to a connection $\Delta_M^{(n)}$– which we call the *exponential direct connection* (eq. (96) in Definition 4.4)– on the the jet prolongation $J^n\,\mathrm{Pair}(M)$ of the pair groupoid of M. In Theorem 4.5 we prove that the exponential direct connection $\Delta_M^{(n)}$ is a jet prolongation of Δ^M and in Theorem 4.6 that the infinitesimal connection of $\Delta_M^{(n)}$ on $J^n\,\mathrm{Pair}(M)$ is the exponential n-th order prolongation $\delta_M^{(n)} : TM \longrightarrow \mathcal{L}(J^n\,\mathrm{Pair}(M)) \cong J^nTM$ (eq. (99)) of the affine connection on M used in [Kol09, §5] to build infinitesimal connections on jet prolongations of groupoids.

A similar construction using an affine connection on the underlying manifold, yields a direct connection $\Gamma^{(n)}$ on the jet prolongation $J^n\mathcal{G}$ of a general Lie groupoid $\mathcal{G}$ from a direct connection Γ on $\mathcal{G}$, see eq. (103) in Definition 4.7, which gives back eq. (96) when $\mathcal{G} = \mathrm{Pair}(M)$. Corollary 4.8 shows that $\Gamma^{(n)}$, which yields an n-th order prolongation of Γ, factorises through $\Delta_M^{(n)}$. In Theorem 4.11, we show that any flat connection on the jet prolongation $J^n\mathcal{G}$ of a Lie groupoid over a flat manifold, factorises through $\Delta_M^{(n)}$.

Direct connections on the frame groupoid $\mathrm{Iso}(J^nE)$ of the jet bundle J^nE of a vector bundle $E \to M$ are of special interest in the context of regularity structures.

Specialising to a direct connection Γ on the frame groupoid $\mathcal{G} = \mathrm{Iso}(E)$ of a vector bundle $E \to M$, the above construction yields a n-th order prolonged direct connection $\Gamma^{(n)}$ on $J^n\mathrm{Iso}(E)$. This in turn induces a direct connection –again denoted by $\Gamma^{(n)}$ with some abuse of notation– on $\mathrm{Iso}(J^nE) \subsetneq J^n\mathrm{Iso}(E)$. We compare it (Proposition 4.18) with another direct connection $\widetilde{\Gamma}^{(n)}$ given by eq. (123) in §4.7, built by means of a local Taylor expansion following the construction in [DDD19, Definition 76]. Unlike the family of direct connections $\Gamma^{(n)}, n \in \mathbf{N}$ on $\mathrm{Iso}(J^nE), n \in \mathbf{N}$, which yields a projective system, the family of direct connections $\widetilde{\Gamma}^{(n)}, n \in \mathbf{N}$ obtained by means of Taylor expansions which are relevant in the context of regularity structures, does not.

Regularity structures are briefly discussed in Section 5, where we propose a notion of *geometric pre-regularity structure* (Definition 5.1) on a vector bundle of finite rank on a manifold M. It offers a geometric framework to host the algebraic data in Hairer's regularity structures on $\mathbb{R}^d$ [H14] and the polynomial regularity structures on a Riemannian manifold built in [DDD19], leaving out the analytic aspects, hence the prefix "pre".

The geometric framework we propose keeps track of the structure group in the form of a groupoid and its action on the vector bundle defined in terms of a direct connection on the frame groupoid of this veector bundle. The underlying geometric structures are given in the projective setup (briefly discussed in Appendix 6), and which is well suited to keep track of the grading inherent to regularity structures and perturbative approaches to quantum field theory. The model space T in Hairer's regularity structure is replaced by a projective limit $E := \varprojlim_\alpha E_\alpha$ of vector bundles $E_\alpha \to M, \alpha \in A$ over a manifold M, indexed by a discrete set A bounded from below. The typical fibre of E is a graded space $T = \oplus_{\alpha\in A} T^\alpha$, the frame groupoid $\mathrm{Iso}(E)$ of E is acted upon by a prounipotent gauge groupoid $\mathcal{G}(P) := \varprojlim_\alpha \mathcal{G}(P_\alpha) \rightrightarrows M$ (as described in eq. (6.8)) and the structure group in Hairer's framework is the structure group of the principal bundle $P = \varprojlim_\alpha P_\alpha \to M$ given as the inverse limit of a projective system of principal bundles underlying the gauge groupoid.

To such a geometric pre-regularity structure, we associate a geometric pre-model, leaving out the analytic requirements for a full fledged model as defined in the context of regularity structures, hence the prefix "pre" in front of "regularity structures". As in [H14], it is given by a pair (Π, Γ): here, Π is a family of maps from the total space E of the vector bundle to a sheaf $\mathcal{D}'_M(-, E^0)$ of vector valued distributions, and Γ is a (not necessarily projective) family of direct connections on the underlying gauge groupoid $\mathcal{G}(P)$. In Proposition 5.3, we express the obstruction $\Pi \circ \Gamma - \Pi$ to the "Γ-invariance of Π" in terms of a curvature term (140) for Γ. In particular, this obstruction vanishes in the flat case.

Polynomial pre-regularity structures discussed in §5.2, bring together the main geometric and analytic ingredients of the paper, namely groupoids equipped with direct connections discussed in Section 2 and direct connections on jet prolongations discussed in Section 3. Theorem 5.2 puts geometric polynomial regularity structures in the general framework of geometric pre-regularity structures. There, the bundle E is the jet bundle of a vector bundle $E^0 \to M$ with the index set given by $A = \mathbb{Z}_{\geqslant 0}$ and E^n is the n-jet prolongation $J^n E^0$ of E^0. The frame bundle of the bundle E therefore involves the frame bundles $\mathrm{Iso}(J^n E^0)$ of jet prolongations of E^0. The pre-model (Π, Γ) is built along the lines of [DDD19, Definition 80], from a Taylor expansion map, using the direct connection on $\mathrm{Iso}(J^n E)$ defined in eq. (123).

The case $E^0 = M \times \mathbb{R}$ corresponds to the polynomial regularity structure in the framework of [H14] if $M = \mathbb{R}^d$ and that of [DDD19] on a Riemannian manifold M. Theorem 5.6 then revisits Dahlqvist, Diehl and Driver's [DDD19] polynomial regularity structures in the language of jet prolonged groupoids with direct connections. Our construction on $J^n E^0$ relates to that of [DDD19] on $\Sigma^n T^* M \otimes E^0$ via the isomorphism $J^n E \simeq \Sigma^n T^* M \otimes E^0$ given by Eq. (125) in §4.7 induced by a connection on M.

Openings. This exploratory paper is a first step towards further possible investigations, one of which would be to transpose the geometric constructions carried out here in the smooth setting to the Hölder setting better suited for the study of sPDEs. Also, direct connections on gauge groupoids viewed as an integrated version of connections on principal bundles, open the road to the study of higher gauge theories.

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1. Lie groupoids and Lie algebroids

Groupoids can be viewed as a fibred generalisation of groups over manifolds or as a generalisation of groups as fibred objects over manifolds. They were introduced by Brandt in [Bra26], who actually introduced what are now called transitive groupoids. Interest in groupoids broadened in the 50's when the notion of category arose, since the invertible elements of a small category form a groupoid. As from then, the use of groupoids was expanded by Ehresmann in various areas of mathematics, including differential geometry. Groupoids have become central tools to host singular structures.

In this section we recall basic facts on Lie groupoids as well as on their actions on fibre bundles, and the main examples needed in the sequel. Our main references are the standard textbooks by K. Mackenzie [MK05], I. Moerdijk and J. Mrčun [MM03] and E. Meinrencken [Me17], and the pedagogical introduction by A. Kumpera [Ku15]. Explicit references are quoted for specific results. We recall the relation between principal bundles and gauge groupoids (Proposition 1.5) and consider reduced frame groupoids obtained as the gauge groupoid of a reduced frame bundle (Proposition 1.14). We finish this section with a short review of (local) bisections which later enter the construction of jet-prolongations of groupoids.

1.1. **Lie groupoids.** A **groupoid** on a manifold M is a set $\mathcal{G}$, whose elements are called **arrows**, together with the following structure maps:

(1) a **source** map $s : \mathcal{G} \longrightarrow M$ and a **target** map $t : \mathcal{G} \longrightarrow M$,
(2) a **multiplication** (or **composition**) $m : \mathcal{G} \times_M \mathcal{G} \longrightarrow \mathcal{G}, \ (\gamma_1, \gamma_2) \mapsto \gamma_1\gamma_2$, defined on the set $\mathcal{G} \times_M \mathcal{G} = \{(\gamma_1, \gamma_2), \ s(\gamma_1) = t(\gamma_2)\}$ of **composable arrows**, assumed to be associative,
(3) a **unit** map $u : M \to \mathcal{G}, \ x \mapsto u(x) =: 1_x$ such that $t(1_x) = s(1_x) = x$ for any $x \in M$ and $1_{t(\gamma)}\,\gamma = \gamma = \gamma\, 1_{s(\gamma)}$ for any $\gamma \in \mathcal{G}$,
(4) an **inversion** $i : \mathcal{G} \longrightarrow \mathcal{G}, \ \gamma \mapsto \gamma^{-1}$ such that $s(\gamma^{-1}) = t(\gamma), \ t(\gamma^{-1}) = s(\gamma)$, $\gamma\,\gamma^{-1} = 1_{t(\gamma)}$ and $\gamma^{-1}\gamma = 1_{s(\gamma)}$.

The induced map $(t, s) : \mathcal{G} \longrightarrow M \times M$ is called the **anchor**. From the axioms it follows that the source and the target are surjective maps, the unit map is injective and the inversion is bijective. The manifold M is called the **base** of the groupoid and can be identified with the **set of units** $u(M) \subset \mathcal{G}$. A groupoid is compactly denoted by $\mathcal{G} \rightrightarrows M$ and the structure maps (s, t, m, u, i) are tacitely understood.

A group G can be seen as a groupoid $G \rightrightarrows *$ on the base manifold given by a point, with trivial source and target maps. Hence groupoids generalise groups.

Given a groupoid $\mathcal{G} \rightrightarrows M$ and points x, y in M, we use the following notations:

- $\mathcal{G}^x := t^{-1}(x)$ for the t-**fibre** of x, with restricted source map $s^x := s_{|\mathcal{G}^x} : \mathcal{G}^x \to M$,
- $\mathcal{G}_y := s^{-1}(y)$ for the s-**fibre** of y, with restricted target map $t_y := t_{|\mathcal{G}_y} : \mathcal{G}_y \to M$,
- $\mathcal{G}^x_y := \mathcal{G}^x \cap \mathcal{G}_y$ for the **fibre** of (x, y), whose arrows are often denoted γ_{xy} (or γ^x_y).

Similarly, for $U, V \subset M$, we set $\mathcal{G}^U := t^{-1}(U)$, $\mathcal{G}_V := s^{-1}(V)$ and $\mathcal{G}^U_V := \mathcal{G}^U \cap \mathcal{G}_V$.

For any $x \in M$, the set $\mathcal{G}^x_x$ is a (non-empty) group with the composition of arrows and unit 1_x, called the **vertex group** (or the **isotropy**) at x. The non emptyness of the set $\mathcal{G}^x_y$ above two distinct points x, y in M defines an equivalence relation: $x \sim y$ if and only if $\mathcal{G}^x_y$ whose equivalence classes are called **orbits** of $\mathcal{G}$. The orbit of a point x in M is the set $\mathcal{O}_x = s^x(\mathcal{G}^x) = t_x(\mathcal{G}_x) \subset M$. The **orbit space** of $\mathcal{G}$, denoted $M/\mathcal{G}$, is the quotient of

M by the relation and gives a foliation of M which is possibly singular. The groupoid $\mathcal{G}$ is **regular** if the orbits have all the same dimension, that is, the foliation is regular, and it is **transitive** if it has a single orbit M. This holds if and only if the anchor map is surjective.

A groupoid $\mathcal{G} \rightrightarrows M$ is a **Lie groupoid** if

(1) it is smooth i.e., if $\mathcal{G}$ and M are smooth manifolds
(2) and the source and target maps are surjective submersions.

This guarantees the following nice properties:

Fact 1.1. [MM03, Theorem 5.4] *Let $\mathcal{G} \rightrightarrows M$ be a Lie groupoid. Then, for any x, y in M:*

(1) The unit set $u(M) \subset \mathcal{G}$, is a submanifold of $\mathcal{G}$, and by assumption the set of composable arrows $\mathcal{G} \times_M \mathcal{G}$ is a manifold (this holds for any smooth groupoid).
(2) The vertex group $\mathcal{G}_x^x$ is a Lie group.
(3) The fibre $\mathcal{G}_y^x$ is a closed submanifold of $\mathcal{G}$ (possibly empty).
(4) The orbit $\mathcal{O}_x = t_x(\mathcal{G}_x) = s^x(\mathcal{G}^x)$ is an immersed submanifold of M and the restricted maps t_x and s^x are both principal $\mathcal{G}_x^x$-bundles on $\mathcal{O}_x$.

Moreover, if $\mathcal{G}$ is transitive then the anchor map is a surjective submersion.

Thanks to these properties, Lie groupoids allow an infinitesimal calculus (via Lie algebroids) analogous to that defined on Lie groups (via Lie algebras), and are suitable to study smooth actions on fibre bundles.

However, one should keep in mind that the vertex groups $\mathcal{G}_x^x$ and $\mathcal{G}_y^y$ over distinct points are not necessarily isomorphic, since the fibre $\mathcal{G}_x^y$ can be empty and the groups can belong to separate connected components of $\mathcal{G}$, even if the base manifold M is connected.

A morphism between two groupoids is a functor between the (category-theoretic) groupoids.

We focus on **morphisms over the identity map** (also called **morphism over** M or **morphism preserving the units**) between two Lie groupoids $\mathcal{G} \rightrightarrows M$ and $\mathcal{G}' \rightrightarrows M$ respectively with source s, s' and target t, t' and the same base manifold M, namely smooth maps $\phi : \mathcal{G} \to \mathcal{G}'$ such that

(1) $s' \circ \phi = s$ and $t' \circ \phi = t$,
(2) $\phi \circ u = u'$,
(3) $\phi(\gamma\,\gamma') = \phi(\gamma)\,\phi(\gamma')$ for any composable γ, γ' in $\mathcal{G}$, and therefore $\phi \circ i = i' \circ \phi$.

It is an **isomorphism** of Lie groupoids if ϕ is a diffeomorphism.

A **subgroupoid** of a Lie groupoid $\mathcal{G} \rightrightarrows M$ is a groupoid $\mathcal{G}' \rightrightarrows M$ together with an injective Lie groupoid morphism $\iota : \mathcal{G}' \hookrightarrow \mathcal{G}$ over M, giving the inclusion.

Examples 1.1. Let M be a smooth manifold.

(1) Given a Lie group G, the cartesian product $M \times G \times M$ defines a Lie groupoid on M, called **trivial groupoid with vertex group** G, with source $s(x, g, y) = y$, target $t(x, g, y) = x$, composition $(x, g, y)(y, h, z) = (x, gh, z)$ induced by the product on G, unit $1_x = (x, 1_G, x)$, where 1_G is the unit on G and inverse $(x, g, y)^{-1} = (y, g^{-1}, x)$ with g^{-1} the inverse of g in G, for any x, y, z in M and g, h in G.

   A Lie groupoid $\mathcal{G} \rightrightarrows M$ is called **trivial** if it is isomorphic to a trivial groupoid. Any trivial Lie groupoid is clearly transitive.
(2) The **pair groupoid** of M is the trivial groupoid with trivial vertex group $G = \{e\}$, namely the cartesian product $\mathrm{Pair}(M) = M \times M$, where arrows are pairs (x, y) of points, with source $s(x, y) = y$, target $t(x, y) = x$, composition $(x, y)\,(y, z) = (x, z)$,

unit $1_x = (x, x)$ and inverse $(x, y)^{-1} = (y, x)$. The diagonal $\Delta_M = \{(x, x), x \in M\}$ of M corresponds to the set of units of the pair groupoid.

For any Lie groupoid $\mathcal{G} \rightrightarrows M$, the anchor map $(t, s) : \mathcal{G} \longrightarrow \mathrm{Pair}(M)$ is a Lie groupoid morphism over M, which maps the units of $\mathcal{G}$ onto the units of $\mathrm{Pair}(M)$, that is, $(t, s)\big(u_{\mathcal{G}}(M)\big) = \Delta_M$. The anchor $(t, s) : \mathcal{G} \longrightarrow M \times M$ is a local fibration.

(3) The **fundamental groupoid** of M is the set $\Pi(M)$ of homotopy classes $[\gamma]$ of continuous paths $\gamma : [0, 1] \to M$ with source $s([\gamma]) = \gamma(0)$, target $t([\gamma]) = \gamma(1)$, partial composition $[\gamma]\,[\tilde{\gamma}] = [\gamma\,\tilde{\gamma}]$ induced by the concatenation of paths $\gamma, \tilde{\gamma} : [0, 1] \to M$ such that $\tilde{\gamma}(1) = \gamma(0)$, and inversion $[\gamma]^{-1} = [\gamma^{-1}]$ induced by the inversion of orientation. Its vertex group at a point x_0 in M is the fundamental group $\pi_1(M, x_0)$, see [MK05, Examples 1.1.1, 1.3.4]. One can show that it is a Lie groupoid with the quotient topology and that it is transitive if and only if M is connected [Me17, Example 1.10].

□

1.2. **Local maps and local morphisms.** In a fibre bundle, an object is *local* if it is defined in an open neighborhood of a base point. For a groupoid, this notion must be adapted to the fact that its very essence is to relate distinct base points: *locality* then means that the points to be related are sufficiently close to one another, wherever they are in the base manifold. This leads to the following definition. Since we shall only be concerned by maps over the identity, we omit specifying it.

Definition 1.2. A **local map** between two (resp. Lie) groupoids $\mathcal{G} \rightrightarrows M$ and $\mathcal{G}' \rightrightarrows M$ over a manifold M is a (resp. smooth) map $\phi : \mathcal{U} \subset \mathcal{G} \longrightarrow \mathcal{G}'$ defined on an open neighbourhood $\mathcal{U}$ of the units $u(M) \subset \mathcal{G}$, which commutes with the source, the target and the units, that is,

(1) $s' \circ \phi = s$ and $t' \circ \phi = t$,
(2) $\phi \circ u = u'$.

If $\mathcal{U} = \mathcal{G}$, we call ϕ a **global map**. To distinguish local from global maps at a glance, we denote local maps by $\phi : \mathcal{G} \ast\!\!\longrightarrow \mathcal{G}$. Local maps between Lie groupoids are assumed to be smooth, unless otherwise specified.

A local map defined on an open set $\mathcal{U}$ restricts to a local map on any open subsets $\mathcal{U}' \subset \mathcal{U}$ containing $u(M) \subset \mathcal{G}$. Following Mackenzie, we call two local maps over M **germ equivalent** if they agree on some neighborhood of $u(M)$. A local map $\phi : \mathcal{G} \ast\!\!\longrightarrow \mathcal{G}'$ is a **local morphism** of (resp. Lie) groupoids if it also preserves compositions, i.e.

(3) $\phi(\gamma\,\gamma') = \phi(\gamma)\,\phi(\gamma')$ for any composable γ, γ' in $\mathcal{U}$ whose product $\gamma\,\gamma'$ lies in $\mathcal{U}$.

In this case, it also preserve inversions (resp. which are smooth), i.e. $\phi(\gamma^{-1}) = \phi(\gamma)^{-1}$ for all $\gamma \in \mathcal{G}$ such that $\gamma\gamma^{-1}$ lies in $\mathcal{U}$. We denote local morphisms by $\phi : \mathcal{G} \circ\!\!\longrightarrow \mathcal{G}'$, as in [MK05, Definition 6.1.6]. A global morphism in this sense is the same as a groupoid morphism of Section 1.1 which we shall therefore simply call morphism.

A local map between two Lie groupoids cannot always be extended to a global one, even if it is a local groupoid morphism. Mackenzie proved in [MK05, Theorem 6.1.10] that this is possible under rather restrictive conditions, namely when $\mathcal{G} \circ\!\!\longrightarrow \mathcal{G}'$ is a local groupoid morphism of locally trivial Lie groupoids over the same base manifold M, if the source-fibres (equivalently,the target-fibres) of $\mathcal{G}$ are connected and simply connected.

1.3. **Locally trivial groupoids.** In practice, the Lie groupoids we are interested in are all gauge groupoids, whose structure is simple to describe. Let $\pi_P : P \longrightarrow M$ be a principal bundle with structure group G acting on the fibres on the right, transitively and without fixed points. The **gauge** or **Atiyah groupoid of** P is the quotient manifold

$$\mathcal{G}(P) = P \times_G P := (P \times P)/\sim \tag{2}$$

under the equivalence relation

$$(p, q) \sim (p\, g, q\, g) \qquad \forall p, q \in P, \quad \forall g \in G,$$

whose arrows are equivalence class of pairs (p, q) in $P \times P$ denoted $[p, q]$, endowed with source and target maps given by the bundle projection, namely

$$s([p, q]) = \pi_P(q) \qquad \text{and} \qquad t([p, q]) = \pi_P(p) \qquad \forall p, q \in P,$$

partial composition

$$[p, q]\, [p', q'] = [p, q' g] \qquad \text{for the unique } g \text{ in } G \text{ such that } q = p' g,$$

defined if $\pi^P(q) = \pi^P(p')$, units $1_x = [p, p]$ for any p in $\pi_P^{-1}(x)$ and inverse $[p, q]^{-1} = [q, p]$.

Example 1.3. (1) Given a Lie group G, the trivial groupoid $M \times G \times M$ is clearly a gauge groupoid for the trivial principal bundle $P = M \times G$. This holds in particular for the pair groupoid $\mathrm{Pair}(M)$ whose structure group G is trivial.
(2) The fundamental groupoid $\Pi(M)$ is a gauge groupoid for the principal bundle given by the universal covering of M [MK05, Example 1.3.4].

It is easy to verify that a gauge groupoid $\mathcal{G}(P)$ is trivial, i.e. isomorphic to $M \times G \times M$, if and only if the underlying principal bundle $P \to M$ is trivial, i.e. isomorphic to $M \times G$. Gauge groupoids are caracterized by a very simple local structure. Let us first fix some terminology.

Definition 1.4. [Ku15, §6, Example d)] [MK05, Definition 1.3.2] Let $\mathcal{G} \rightrightarrows M$ be a Lie groupoid.

- $\mathcal{G}$ is **locally trivial** if for any x in M there exists an open neighborhood U of x in M such that $\mathcal{G}_U^U$ is isomorphic to the trivial groupoid $U \times \mathcal{G}_x^x \times U$.
- $\mathcal{G}$ admits a **section atlas** if there exists a point $x_0 \in M$ and a collection of local sections of t_{x_0}, i.e. an open covering $\{U_\alpha\}$ of M and smooth maps $\sigma_\alpha : U_\alpha \to \mathcal{G}_{x_0}^{U_\alpha}$ such that $t_{x_0} \circ \sigma_\alpha = \mathrm{Id}_{U_\alpha}$, called **local decomposing maps**. This implies that the restriction $t_{x_0} : \mathcal{G}_{x_0} \to M$ is a surjective submersion, which is not *a priori* ensured in a Lie groupoid.
- If $\mathcal{G}$ admits a section atlas $\{\sigma_\alpha\}$ based at x_0, the **transition functions** are the maps $g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathcal{G}_{x_0}^{x_0}$ given by $g_{\alpha\beta}(x) = \sigma_\alpha(x)^{-1}\sigma_\beta(x)$.

We now assume that the base manifold M is connected.

Proposition 1.5. [Ku15, Lemma 2] [MK05, Propositions 1.3.3 and 1.3.5] [Me17, Theorem 3.10] *Let $\mathcal{G} \rightrightarrows M$ be a Lie groupoid on a connected manifold. The following assertions are equivalent:*

(1) $\mathcal{G}$ is a gauge groupoid.
(2) $\mathcal{G}$ is transitive (then by Theorem 1.1 the surjective anchor map is a submersion).
(3) $\mathcal{G}$ is locally trivial.
(4) $\mathcal{G}$ admits a section atlas.

(5) *The anchor map* $(t,s):\mathcal{G}\to M\times M$ *is a locally trivial fibre bundle with fibre* $\mathcal{G}_{x_0}^{x_0}$ *for a point* x_0 *in* M*, and structure group* $\mathcal{G}_{x_0}^{x_0}\times\mathcal{G}_{x_0}^{x_0}$ *acting on the left on the fibre by* $(g',g)\cdot h = g'hg^{-1}$*, for any* g',g,h *in* $\mathcal{G}_{x_0}^{x_0}$*.*
(6) *(Assuming* M *is connected) The source map* $s:\mathcal{G}=\bigcup_{x\in M}\mathcal{G}_x\to M$ *is a locally trivial fibration in principal* G*-bundles on* M*, with* $G=\mathcal{G}_{x_0}^{x_0}$ *for any choice of* $x_0\in M$*. The principal* $\mathcal{G}_x^x$*-bundle* $t:\mathcal{G}_x\to M$ *is called the* **vertex bundle** *of* $\mathcal{G}$ *at the point* x [MK05, §1.3].

The proof of these equivalences is based on the fact that one can define local sections for any fibration by the implicit function theorem.

Example 1.6 (Frame groupoids)**.** Any vector bundle $\pi_E : E \longrightarrow M$ of rank r is associated to a principal $GL_r(\mathbb{R})$-bundle $\pi : FE \longrightarrow M$, called the **frame bundle of** E, with fibre $F_xE = \mathrm{Iso}\,(\mathbb{R}^r, E_x)$ above x in M given by the set of linear isomorphisms $\varphi^x:\mathbb{R}^r\to E_x$ (the **frames** of E_x), and projection $\pi(\varphi^x)=x$. The gauge groupoid of FE is called the **frame groupoid of** E and denoted by

$$\mathrm{Iso}(E) := \mathcal{G}(FE) \rightrightarrows M. \tag{3}$$

Its arrows are the linear isomorphisms $\varphi_y^x : E_y \to E_x$ between fibres of E, and the whole groupoid can be described as the set

$$\mathrm{Iso}(E) = \bigcup_{x,y\in M} \mathrm{Iso}\,(E_y, E_x)$$

with source $s(\varphi_x^y) = y$, target $t(\varphi_y^x) = x$ and partial composition $\varphi_y^x\,\psi_z^y : E_z \to E_x$ given by the usual composition of linear maps $\psi_z^y : E_z \to E_y$ and $\varphi_y^x : E_y \to E_x$. Since it is a gauge groupoid, the frame groupoid is a transitive Lie groupoid.

□

1.4. **Groupoid actions.** Let $\mathcal{G}\rightrightarrows M$ be a Lie groupoid and Q a smooth manifold with a smooth map $\varphi: Q\to M$. We call

$$\mathcal{G}\times_M Q := \{(\gamma,q)\in\mathcal{G}\times Q,\ s(\gamma)=\varphi(q)\} = \bigcup_{q\in Q}\mathcal{G}_{\varphi(q)}\times\{q\}.$$

resp.

$$Q\times_M \mathcal{G} := \{(q,\gamma)\in Q\times\mathcal{G},\ t(\gamma)=\varphi(q)\} = \bigcup_{q\in Q}\{q\}\times\mathcal{G}_{\varphi(q)}.$$

the set of **composable** pairs (γ,q) in $\mathcal{G}\times Q$, resp. (q,γ) in $Q\times\mathcal{G}$.

A left, rep. right **action** of $\mathcal{G}$ on Q is given by a surjective submersion $\varphi : Q\to M$, together with an action map [MK05, Definition 1.6.1], [MM03, §5.3. Semi-direct products]

$$\begin{array}{rcl} \mathcal{G}\times_M Q & \longrightarrow & Q \\ (\gamma,q) & \longmapsto & \gamma\cdot q \end{array} \tag{4}$$

resp.

$$\begin{array}{rcl} Q\times_M \mathcal{G} & \longrightarrow & Q \\ (q,\gamma) & \longmapsto & q\cdot\gamma \end{array} \tag{5}$$

such that $\varphi(\gamma\cdot q) = t(\gamma)$, resp. $\varphi(q\cdot\gamma)=s(\gamma)$ and which is compatible with the groupoid composition in the sense that $1_{\varphi(q)}\cdot q = q$ for any q in Q and

$$\gamma\cdot(\gamma'\cdot q) = (\gamma\,\gamma')\cdot q \quad \text{resp.} \quad (q\cdot\gamma')\cdot\gamma = q\cdot(\gamma'\,\gamma)$$

for any composable arrows γ, γ' in $\mathcal{G}$ and for any q in Q composable with γ'.

For such an action, one can form the **semi-direct product groupoid** of the $\mathcal{G}$-action, or **translation groupoid** $\mathcal{G} \ltimes Q$, resp. $Q \rtimes \mathcal{G}$, which indeed defines a Lie groupoid over M.

Recall that the canonical projection $\pi : Q \to M$ of a locally trivial fibration is a surjective submersion.

If $\pi : Q \to M$ is a fibre bundle, we take $\varphi = \pi$ and denote by $\rho_\gamma : Q \to Q$ the map $q \mapsto \gamma \cdot q$ induced by the left action. If $\pi : E \to M$ is a vector bundle, the action ρ of $\mathcal{G}$ on E is called **linear** whenever for any γ in $\mathcal{G}$ the map $\rho_\gamma : E \to E$ acts linearly on the fibres. In other words, linear actions of $\mathcal{G}$ on E are given by groupoid morphisms $\mathcal{G} \to \mathrm{Iso}(E)$ over M [MK05, Definition 1.7.1] [Me17, §5.3]. In this case, the bundle E is also called a **linear representation** of the groupoid $\mathcal{G}$. As usual, the representation is called **faithful** if the corresponding groupoid morphism $\mathcal{G} \to \mathrm{Iso}(E)$ is injective.

Examples 1.7. Let $E \to M$ be a vector bundle on a manifold M.

(1) A faithful linear action of the pair groupoid $\mathrm{Pair}(M) \rightrightarrows M$ on E, that is, an injective morphism $\mathrm{Pair}(M) \hookrightarrow \mathrm{Iso}(E)$ of groupoids over M, is equivalent to a global trivialization of E [Me17, §5.3].

(2) The frame groupoid $\mathrm{Iso}(E) \rightrightarrows M$ has a natural faithful linear representation on E given by the **evaluation**

$$\mathrm{ev} : \mathrm{Iso}(E) \times_M E \to E, \ (\varphi^x_y, a_y) \mapsto \varphi^x_y(a_y), \tag{6}$$

where the isomorphims $\varphi^x_y : E_y \to E_x$ applied to $a_y \in E_y$ gives an element $\varphi^x_y(a_y)$ in E_x.

(3) If the structure group of the bundle E reduces to the group G, let $P \to M$ be the associated principal G-bundle, with gauge groupoid $\mathcal{G}(P) \rightrightarrows M$. Then $\mathcal{G}(P)$ acts linearly on E, with action $\rho : \mathcal{G}(P) \times_M E \to E$ given by the composition of $\rho_E : \mathrm{Iso}(E) \times_M E \to E$ and the map $\iota : \mathcal{G}(P) \hookrightarrow \mathrm{Iso}(E)$ of Proposition 1.14.

If $E \cong P \times_G V$ if a vector bundle associated with a principal G-bundle $\to M$, we consider classes $[r, v]$ in $P \times_G V$ of elements a in E with $\pi_E(a) = \pi_P(r)$. The set of composable elements is

$$\mathcal{G}(P) \times_M E = \{([p, q], [r, v]), \ \pi_P(q) = \pi_P(r)\}$$

and the action ρ of $\mathcal{G}(P)$ on E is given by

$$\rho([p, q])([r, v]) = [p, g\, v],$$

where g in G is the unique group element such that $r = q\, g$. A linear action of the fundamental groupoid $\Pi(M) \rightrightarrows M$ on a E is equivalent to a flat connection on E [Me17, §5.3].

□

1.5. **Lie algebroids.** A **Lie algebroid** on a manifold M is a vector bundle $q : A \to M$ endowed with a Lie bracket $[\ ,\ ]_A \colon \Gamma(A) \times \Gamma(A) \to \Gamma(A)$ on its space of sections (that is, an antisymmetric $\mathbb{R}$-bilinear map satisfying the Jacobi identity) together with a vector bundle map $a \colon A \to TM$ over M called the **anchor**, or equivalently a $C^\infty(M)$-linear map among sections, which satisfies the Leibniz rule

$$[X, f\, Y]_A = f\, [X, Y]_A + a(X)(f)\, Y$$

for any X, Y, Z in $\Gamma(A)$ and f in $C^\infty(M)$, that is, the bracket $[\ ,\ ]_A$ is a derivation. It then follows that the anchor is a morphism of Lie algebras from $C^\infty(M, A)$ to the space of vector fields on M, i.e. $a([X, Y]_A) = [a(X), a(Y)]$. The Lie algebroid A is **regular** if a has constant rank, and it is **transitive** if a is fibrewise surjective.

A **morphism** between two Lie algebroids $A \to M$ and $A' \to M$ on a manifold M is a morphism of vector bundles $\phi : A \to A'$ which commutes with the anchors and preserves the brackets. Lie algebroids together with morphisms of Lie algebroids build the category **L** of Lie algebroids.

Example 1.8. If $\pi : P \to M$ is a principal G-bundle, the group G acts on the tangent bundle $TP \to P$ with action given by the map $d\rho_g : TP \longrightarrow TP$ tangent to the action $\rho_g : P \to P,\ p \mapsto p\,g$, for any g in G. The quotient bundle

$$A(P) := TP/G \to M \tag{7}$$

by this action is a Lie algebroid, called the **Atiyah algebroid** of P, with anchor $a : A(P) \to TM$ induced by $d\pi : TP \to TM$ (therefore fibrewise surjective). In fact, the space of sections $C^\infty(M, A(P))$ coincides with that of G-invariant vector fields on P and therefore it is closed under the Lie bracket of vector fields. The quotient map $\chi : TP \to TP/G = A(P)$ is a fibrewise isomorphism between the two bundles over different base manifolds.

Being transitive, the Atiyah algebroid fits into a short exact sequence of Lie algebroids over M, called the **Atiyah sequence**,

$$0 \longrightarrow \mathrm{Ker}(a) \cong (P \times \mathfrak{g})/G \longrightarrow A(P) = TP/G \longrightarrow TM \longrightarrow 0, \tag{8}$$

obtained by differentiating the G-equivariant exact sequence of right G-spaces

$$0 \longrightarrow P \times G \cong P \times_M P \longrightarrow P \times M \longrightarrow P \times M \longrightarrow 0,$$

where G acts on $P \times G$ by $(p, g) \cdot h = (ph, h^{-1}gh)$ for p in P and g, h in G, and then taking the quotient by G, cf. [MK05, §3.2], [MM03, §6.4].

□

1.6. **Lie algebroid of a Lie groupoid.** Lie algebroids are fibred analogues of Lie algebras over a manifold, and play for Lie groupoids the role that Lie algebras play for Lie groups. They were introduced by J. Pradines in [Pra67] and are nowadays used to study foliations [MM03], Poisson geometry [Ma06] and sigma models in string theory [BKS05]. There are many interesting examples of Lie algebroids which go beyond our scope, cf. [MK05], [Me17] and [MM03]. We are mainly concerned with the Lie algebroid defined by the tangent space of a Lie groupoid at the units, thanks to the properties listed in Theorem 1.1.

We follow [MK05, §3.5] or [MM03, §4.1][1] and consider the vector subbundle of the tangent bundle $T\mathcal{G} \to \mathcal{G}$ built from the tangent spaces to the source-fibres, namely

$$T^s\mathcal{G} := \bigcup_{\gamma \in \mathcal{G}} T_\gamma \mathcal{G}_{s(\gamma)} \longrightarrow \mathcal{G}.$$

The **Lie algebroid** of $\mathcal{G}$ is the pull-back of $T^s\mathcal{G} \subset T\mathcal{G}$ along the embedding $u : M \to \mathcal{G}$, that is, the collection of tangent spaces of the source-fibres at their units

$$\mathcal{L}(\mathcal{G}) := u^*(T^s\mathcal{G}) = \bigcup_{x \in M} T_{1_x}\mathcal{G}_x.$$

[1] An equivalent definition of the Lie algebroid $\mathcal{LG}$ is given by [Me17, §9.2] or again [MM03, §4.1] as the *normal bundle* $T\mathcal{G}_{|u(M)}/Tu(M)$ of M in $\mathcal{G}$ (cf. [Me17, §8.2]).

The bundle projection onto M is given by the source, which is constant on the fibres $T_{1_x}\mathcal{G}_x$ (and equal to x) since $T^s\mathcal{G}$ coincides with the kernel $\mathrm{Ker}(ds)$ of the tangent map $ds : T\mathcal{G} \to TM$. The anchor is given by the composition $a = dt \circ i : \mathcal{L}(\mathcal{G}) \to TM$, where $Dt : T\mathcal{G} \to TM$ is the tangent map of the target and $i : u^*(T^s\mathcal{G}) \to T^s\mathcal{G} \subset T\mathcal{G}$ is the natural map on the pull-back. The Lie bracket on the sections $C^\infty(M, \mathcal{L}(\mathcal{G}))$ is induced via right-translation by that of vector fields on $\mathcal{G}$, which must be proven to be closed among right-invariant $T^s\mathcal{G}$-valued vector fields. By construction, it satisfies the requirements of Lie algebroids.

Remark 1.9. An alternative definition consists in swapping source and target, taking tangent spaces to the target-fibres, leading to an isomorphic Lie algebroid whose Lie-bracket is induced via left-translation by that of vector fields on $\mathcal{G}$ (see e.g. [MK05, The symmetric construction in §3.5]). In this alternative approach, one considers the pull-back of $T^t\mathcal{G} \subset T\mathcal{G}$ along the embedding $u : M \to \mathcal{G}$, that is, the collection of tangent spaces of the target-fibres at their units and defines $\mathcal{L}'(\mathcal{G}) := u^*(T^t\mathcal{G}) = \bigcup_{x\in M} T_{1_x}\mathcal{G}^x$. The bundle projection onto M is given by the target map, which is constant on the fibres $T_{1_x}\mathcal{G}^x$ and the anchor is given by the composition $a = ds \circ i : \mathcal{L}'(\mathcal{G}) \to TM$, where $ds : T\mathcal{G}' \to TM$ is the tangent to the source map. There is an isomorphism of vector bundles $\mathcal{L}(\mathcal{G}) \simeq \mathcal{L}'(\mathcal{G})$, and the Lie bracket $[\cdot,\cdot]'$ on the space $C^\infty(M, \mathcal{L}'(\mathcal{G}))$ induced via left-translation relates to the Lie bracket $[\cdot,\cdot]$ on $C^\infty(M, \mathcal{L}(\mathcal{G}))$ by $[\cdot,\cdot]' = -[\cdot,\cdot]$.

If $\mathcal{G}$ is a locally trivial groupoid (that is, a gauge groupoid, see Theorem **??**), then $\mathcal{L}(\mathcal{G})$ is a transitive Lie algebroid. The converse holds true if M is connected [MK05, Corollary 3.5.18].

Example 1.10. The Lie algebroid of the gauge groupoid $\mathcal{G}(P)$ built from a principal bundle $P \to M$ is isomorphic to the Atiyah algebroid of P [Me17, Example 9.5 (c)], i.e.

$$\mathcal{L}(\mathcal{G}(P)) \cong A(P). \tag{9}$$

□

Examples 1.11. Lie algebroids of gauge groupoids give rise to various explicit examples:

(1) The Lie algebroid of the pair groupoid $\mathrm{Pair}(M) \rightrightarrows M$ is isomorphic to the tangent bundle $TM \to M$ equipped with the vector field brackets and the identity $TM \to TM$ as anchor, i.e. $\mathcal{L}(\mathrm{Pair}(M)) \cong TM$ [MK05, Example 3.5.11].

(2) So is the Lie algebroid of the fundamental groupoid $\Pi(M)$ isomorphic to the tangent bundle, $\mathcal{L}(\Pi(M)) \cong TM$ [Bro88].

(3) If $P = M \times G$ is the trivial principal G-bundle, and $\mathcal{G}(P) = M \times G \times M$, then $\mathcal{L}(M \times G \times M) \cong A(M \times G) \cong TM \oplus (M \times \mathfrak{g})$, where $\mathfrak{g} = \mathrm{Lie}(G)$, with anchor given by the projection to TM and bracket

$$[X + v, Y + w] = [X, Y]_{\mathfrak{X}(M)} + \big(X(w) - Y(v) + [v, w]_{\mathfrak{g}}\big),$$

where $X, Y \in \mathfrak{X}(M)$ and $v, w \in M \times \mathfrak{g}$ [MK05, Example 3.5.13]. When $G = \{1\}$, this gives back $\mathcal{L}(\mathrm{Pair}(M)) \cong TM$.

(4) [MK05, §3.4] Let us specialise to the case when $P = FE$ is the frame bundle of a vector bundle $\pi : E \to M$ whose gauge groupoid $\mathcal{G}(FE) \simeq \mathrm{Iso}(E)$ is the frame groupoid of E. Its Lie algebroid $\mathcal{L}(\mathrm{Iso}(E)) = T^{\mathrm{lin}}E$ consists of **linear vector fields** (ξ, X) of the bundle $TE \to E$ i.e., pairs (ξ, X_ξ), where $\xi : E \to TE$ is a vector field on E over a vector field $X_\xi : M \to TM$. Equivalently, ξ has a local flow given by a bundle morphism $F_y^\xi : E \to E$ over a local flow f_t^X of X_ξ on M. The anchor map

$a : T^{\mathrm{lin}}E \to TM$ sends the vector field ξ on E to the vector field $X_\xi = d\pi(\xi)$ on M and its kernel is the bundle $T^{\mathrm{lin,v}}E$ of vertical vector fields on E which are linear, i.e. $\xi(e) = (e, \mathbf{X}(\mathbf{e}))$ for any e in E, where $\mathbf{X} : \mathbf{E} \to \mathbf{E}$ is a vector bundle morphism.

The Lie algebroid $T^{\mathrm{lin}}E$ of $\mathrm{Iso}(E)$ coincides with the Atiyah algebroid $A(FE)$ of FE and we have the exact sequence

$$0 \longrightarrow T^{\mathrm{lin,v}}E \longrightarrow T^{\mathrm{lin}}E \xrightarrow{a} TM \longrightarrow 0. \tag{10}$$

(5) [MK05, Example 3.3.4], [MK05, Theorem 3.6.6] The Lie algebroid $\mathcal{L}(\mathrm{Iso}(E))$ of the frame groupoid $\mathrm{Iso}(E)$ is isomorphic to the bundle $\mathrm{Der}(E) \to M$ of linear derivations on E defined as follows.

A **linear derivation** on E at a point x in M [ETV19, §2.1], is an $\mathbb{R}$-linear map $D_x : C^\infty(M, E) \to E_x$ for which there is a vector ξ_{D_x} in T_xM such that

$$D_x(\lambda f) = \xi_{D_x}(\lambda) f(x) + \lambda(x)\, D_x f \quad \forall \lambda \in C^\infty(M), \forall f \in C^\infty(M, E). \tag{11}$$

Let $\mathrm{Der}_x(E)$ be the linear space of linear derivations on E at a point x, then

$$\mathrm{Der}(E) = \bigcup_{x \in M} \mathrm{Der}_x(E) \to M \tag{12}$$

forms a vector bundle over M, called the bundle of linear derivations on E. Its sections are first order differential operators $D : C^\infty(M, E) \to C^\infty(M, E)$ for which there exists a vector field $\xi_D \in \mathfrak{X}(M)$ on M, such that eq. (11) holds for every x in M, setting $D\sigma(x) := D_x(\sigma)$ and $\xi_D(x) := \xi_{D_x}$. The Lie bracket on $\mathrm{Der}(E)$ is the commutator bracket of operators and the anchor is the symbol map $\mathrm{Der}_x(E) \ni D_x \longmapsto X_{D_x} \in T_xM$. Its kernel is isomorphic to $\mathrm{End}(E_x)$ and the Atiyah sequence in eq. (8) reads

$$0 \longrightarrow \mathrm{End}(E) \longrightarrow \mathrm{Der}(E) \xrightarrow{a} TM \longrightarrow 0. \tag{13}$$

We follow [MK05, Proposition 3.4.4] and[L-GV, Example 2.3] to describe the isomorphism

$$\rho : \mathcal{L}(\mathrm{Iso}(E)) \xrightarrow{\simeq} \mathrm{Der}(E), \tag{14}$$

which sends a linear vector field (ξ, X) in $T^{\mathrm{lin}}E$ to a linear derivation D_ξ in $\mathrm{Der}(E)$ defined as follows. For any section f of E and for any section ϕ of E^* [MK05, Eq. (27) p.115]

$$\langle \phi, D_\xi(f) \rangle = X\langle \phi, f \rangle - \xi(\ell_\phi)(f). \tag{15}$$

Here ℓ_ϕ is the fibrewise-linear function on E corresponding to the section ϕ so that $\ell_\phi(f) = \langle \phi, f \rangle$ and $\xi(\ell_\phi)$ is again a fibrewise-linear function on E since linear vector fields preserve the subspace of such functions.

In the remaining part of this paragraph, we consider global morphisms.

Let $P \to M$ and $P' \to M$ be two principal bundles over M with structure groups respectively G and G' and let $\mathcal{G}(P) \rightrightarrows M$ and $\mathcal{G}(P) \rightrightarrows M$ the corresponding gauge groupoids as defined by eq. (2).

(1) Given a morphism $\varphi_0 : G \to G'$ of Lie groups and a morphism $\varphi : P \to P'$ of principal bundles over the identity which is φ_0-equivariant, that is, $\varphi(pg) = \varphi(p)\varphi_0(g)$, the map $\mathcal{G}(\varphi) : P \times_G P \to P' \times_{G'} P'$ given by

$$\mathcal{G}(\varphi)\big([p, q]\big) = [\varphi(p), \varphi(q)], \qquad p, q \in P, \tag{16}$$

defines a groupoid morphism

$$(\mathcal{G}(\mathbf{P}) \rightrightarrows M) \xrightarrow{\mathcal{G}(\varphi)} (\mathcal{G}(\mathbf{P}') \rightrightarrows M')\,.$$

The map $\mathcal{G}(\varphi)$ is well defined since $p' = p\,g$ and $q' = q\,g$ for any $g \in G$ implies that $[\varphi(p'), \varphi(q')] = [\varphi(p)\,\varphi_0(g), \varphi(q)\,\varphi_0(g)] = [\varphi(p), \varphi(q)]$, which clearly defines a groupoid morphism.

(2) Conversely, a morphism of Lie groupoids $\phi : \mathcal{G} \to \mathcal{G}'$ over the identity on M induces by restriction, a morphism $\varphi : P := \mathcal{G}_{x_0} \to \mathcal{G}'_{x_0} =: P'$ of principal bundles and a group morphism $\varphi_0 : G := \mathcal{G}_{x_0}^{x_0} \to (\mathcal{G}')_{x_0}^{x_0} =: G'$, giving rise to a φ_0-equivariant morphism $\varphi : P \to P'$, such that $\mathcal{G}(\varphi) = \phi$.

1.7. **The Lie functor.** Just as the tangent map at the identity of a morphism of Lie groups induces a Lie algebra morphism between the corresponding Lie algebras, there exists a tangent map at the units of a local morphism between Lie groupoids which gives rise to a Lie algebroid morphism. More generally, one can differentiate at the units any local map between Lie groupoids, and obtain its infinitesimal part along the diagonal.

Consider a local map ϕ over the identity between two Lie groupoids $\mathcal{G} \rightrightarrows M$ and $\mathcal{G}' \rightrightarrows M$, defined on $\mathcal{U} \subset \mathcal{G}$. For any $x \in M$, we denote by $D\phi|_x$ the **(target)-differential of ϕ at the unit** $1_x = u(x) \in \mathcal{U}$ given by

$$D\phi|_x(\dot\gamma(0)) = \frac{d}{dt}\phi\big(\gamma(t)\big)|_{t=0},$$

where $\gamma : (-\varepsilon, \varepsilon) \to \mathcal{U} \cap \mathcal{G}_x$ is any smooth curve living in the source-fibre of x such that $\gamma(0) = 1_x$.

Lemma 1.12. *The differential at the units of a local map $\phi : \mathcal{G} \ast\!\!\to \mathcal{G}'$ between two Lie groupoids $\mathcal{G} \rightrightarrows M$ and $\mathcal{G}' \rightrightarrows M'$ induces a vector bundle morphism*

$$D\phi|_M : \mathcal{L}\mathcal{G} \longrightarrow \mathcal{L}\mathcal{G}'$$

between their associated Lie algebroids. If ϕ is a local groupoid morphism, then $D\phi|_M$ is a Lie algebroid morphism.

Proof. By assumption, the differential $D\phi$ is defined on tangent vector fields to the source fibres at the units, which span the fibres of the vector space $\mathrm{Ker}(Ds) = \mathcal{L}\mathcal{G}$. Since ϕ preserves the source and the target, by Definition 1.2 (1), the map $D\phi$ maps $\mathrm{Ker}(Ds)$ to $\mathrm{Ker}(Ds')$ and gives a vector bundle morphism $D\Phi|_M : \mathcal{L}\mathcal{G} \to \mathcal{L}\mathcal{G}'$. The second assertion is proved in [MK05, §3.5, (40) and (41)]. □

Recall that if G and G' are two Lie groups with Lie algebras respectively $\mathfrak{g}$ and $\mathfrak{g}'$, a Lie algebra morphism $\mathfrak{g} \to \mathfrak{g}'$ integrates to a Lie group morphism $G \to G'$ if G is connected and simply connected. The same does *not* hold true for morphisms of Lie algebroids, since those do not always integrate to a morphism of Lie groupoids; the obstruction to integrability is studied in [CF03].

Proposition 1.13. *Let $\mathcal{G} \rightrightarrows M$ and $\mathcal{G}' \rightrightarrows M$ be two locally trivial Lie groupoids on the same base manifold, and let $\varphi\colon \mathcal{L}(\mathcal{G}) \to \mathcal{L}(\mathcal{G}')$ a morphism of Lie algebroids.*

(1) [MK05, Theorem 6.2.4] *If the source-fibres of $\mathcal{G}$ are connected and simply connected, the morphism φ integrates to a global groupoid morphism $\phi : \mathcal{G} \to \mathcal{G}'$, that is, $D\phi|_M = \varphi$.*

(2) [MK05, Theorem 6.2.3] *In general, φ integrates to a local groupoid morphism $\phi : \mathcal{G} \circ\!\!\rightarrow \mathcal{G}'$, in the sense that $D\phi|_M$ coincides with φ on some open subset U in M, in which case ϕ is defined on an open neighborhood of the diagonal in $U \times U$. Two such integrated local morphisms are germ equivalent, i.e. they coincide on a neighborhood of the diagonal.*

To conclude, we have shown that the map $\mathcal{L} : \mathcal{G} \to \mathrm{Lie}(\mathcal{G})$ is functorial, justifying the terminology "Lie functor".

1.8. **Groupoid reductions.** Let $\mathcal{G} \rightrightarrows M$ and $\mathcal{H} \rightrightarrows M$ be two Lie groupoids on M. An injective groupoid morphism $I : \mathcal{G} \hookrightarrow \mathcal{H}$ induces an injective morphism of the vertex groups $\iota_x : \mathcal{G}_x^x \hookrightarrow \mathcal{H}_x^x$ for any $x \in M$. Consequently, the principal $\mathcal{G}_x^x$-bundle $t_x : \mathcal{G}_x \to M$ (resp. $s^x : \mathcal{G}^x \to M$) reduces to the principal $\mathcal{H}_x^x$-bundle $t_x : \mathcal{H}_x \to M$ (resp. $s^x : \mathcal{H}^x \to M$). If the map I exists, we shall say that the groupoid $\mathcal{H}$ **reduces** to $\mathcal{G}$ (see also [MK05, Definition 1.6.22]). The Lie algebroid $\mathcal{L}(\mathcal{H})$ then reduces as a vector bundle to the Lie algebroid $\mathcal{L}(\mathcal{G})$ [MK05, Definition 3.3.21].

For gauge groupoids, it follows from the above discussion that a necessary condition for a gauge groupoid $\mathcal{G}(Q) \rightrightarrows M$ to reduce to a gauge groupoid $\mathcal{G}(P) \rightrightarrows M$ is that the underlying principal bundle $Q \to M$ reduces to the principal bundle $P \to M$ [KN14, Chapter I, §5, p.53]. The next proposition shows the equivalence of the two reductions.

Proposition 1.14. *Let $P \to M$ and $Q \to M$ be two principal bundles on M.*

(1) *The gauge groupoid $\mathcal{G}(Q)$ reduces to $\mathcal{G}(P)$ if and only if the underlying principal bundle Q reduces to P. The reduction is given by the injective groupoid morphism $\mathcal{G}(\iota) : \mathcal{G}(P) \hookrightarrow \mathcal{G}(Q)$ induced by the injective morphism $\iota : P \hookrightarrow Q$ of principal bundles.*
(2) *In particular, the frame groupoid $\mathrm{Iso}(E)$ of a real vector bundle $E \to M$ of rank r reduces to the gauge groupoid $\mathcal{G}(P)$ of a principal G-bundle, for a subgroup G of $GL_r(\mathbb{R})$, if and only if the frame bundle FE reduces to P.*

Proof. The second statement follows from the first one applied to the frame bundle $Q = F(E)$ with structure group $H = GL_r(\mathbb{R})$.

An injective morphism $\iota : Q \to P$ of a principal H-bundle to a principal G-bundle over M is an injective smooth map which preserves the fibres and such that $\iota(p\,g) = \iota(p)\,g$ for any g in G [KN14, Proposition I.5.3]. This map induces an injective morphism $\mathcal{G}(\iota) : \mathcal{G}(P) \longrightarrow \mathcal{G}(Q)$ of Lie groupoids over M defined in eq. (16).

Let us prove that it is injective: if $[\iota(p_1), \iota(p_2)] = [\iota(p_1'), \iota(p_2')]$, the elements p_1, p_1' belong necessarily to the same fibre of P and the same for p_2, p_2', because ι preserves the source and the target. On the one hand, there exist g_1, g_2 in G such that $p_1' = p_1\,g_1$ and $p_2' = p_2\,g_2$ since the G-action on P is transitive, from which it follows that $\iota(p_1') = \iota(p_1)\,g_1$ and $\iota(p_2') = \iota(p_2)\,g_2$. On the other hand, there is an element $h \in H$ such that $\iota(p_1') = \iota(p_1)\,h$ and $\iota(p_2') = \iota(p_2)\,h$. Since the H-action on Q is free, we have $h = g_1$ and $h = g_2$. Thus, h lies in G and $g_1 = g_2$. Consequently, $[p_1', p_2'] = [p_1\,h, p_2\,h] = [p_1, p_2]$. □

1.9. **Bisections.** Let $\mathcal{G} \rightrightarrows M$ be a Lie groupoid. A **local bisection** of $\mathcal{G}$ is a smooth local section $\sigma : M \twoheadrightarrow \mathcal{G}$ of the source map s defined on an open subset $U \subset M$, that is, $s \circ \sigma = \mathrm{Id}_U$, whose composition with the target map t is a diffeomorphism $\varphi_\sigma = t \circ \sigma$ between U and an open subset $\varphi_\sigma(U) \subset M$ (which might not intersect U).

If $U = \varphi_\sigma(U) = M$, we call σ a **global bisection** or simply a **bisection**. The space $\mathcal{B}(\mathcal{G})$ of global sections of $\mathcal{G}$ is a group with the following operations:

(1) multiplication of σ and σ' given by $(\sigma' \bullet \sigma)(x) = \sigma'(\varphi_\sigma(x))\,\sigma(x)$ for any $x \in M$, which corresponds to a semidirect product law on pairs (φ_σ, σ).
(2) unit bisection given by the unit map $u : M \to \mathcal{G}$ of the groupoid $\mathcal{G}$,
(3) inverse σ^{-1} given by $\sigma^{-1}(x) = \big(\sigma(\varphi_\sigma^{-1}(x))\big)^{-1}$ for any $x \in M$, where $\varphi_{\sigma'}^{-1}$ denotes the inverse diffeomorphism of M while the external $(\ )^{-1}$ denotes the inverse in the groupoid $\mathcal{G}$.

Moreover, the map

$$\varphi : \mathcal{B}(\mathcal{G}) \to \mathrm{Diff}(M),\ \sigma \mapsto \varphi_\sigma = t \circ \sigma \tag{17}$$

is a group homomorphism, because $\sigma' \bullet \sigma = (\sigma' \circ \varphi_\sigma)\sigma$, seen as a pointwise groupoid multiplication, and therefore

$$\varphi_{\sigma' \bullet \sigma} = t \circ \big((\sigma' \circ \varphi_\sigma)\sigma\big) = t \circ (\sigma' \circ \varphi_\sigma) = (t \circ \sigma') \circ \varphi_\sigma = \varphi_{\sigma'} \circ \varphi_\sigma.$$

The space of local sections $\mathcal{B}_{\mathrm{loc}}(\mathcal{G})$ is a pseudo-group with the above operations, because the product $\sigma' \bullet \sigma$ of two local sections defined respectively on two open sets U' and U is defined if and only if the target space $\varphi_\sigma(U)$ of σ has a non-empty intersection with the source space U' of σ'. In [SW15, Theorem 2.8] the authors show that for is a locally convex and locally metrisable Lie groupoid over M which admits an adapted local addition. The map in eq. (17) descends to a pseudo-group homomorphism

$$\varphi : \mathcal{B}_{\mathrm{loc}}(\mathcal{G}) \to \mathrm{Diff}_{\mathrm{loc}}(M),\ \sigma \mapsto \varphi_\sigma = t \circ \sigma$$

from local bisections of $\mathcal{G} \rightrightarrows M$ to the set of diffeomorphisms between open subsets of M, that we improperly denote by $\mathrm{Diff}_{\mathrm{loc}}(M)$.

Example 1.15. [Me17, Examples 3.5] For a gauge groupoid, global bisections are in bijection with the automorphisms of the underlying principal bundle: $\mathcal{B}(\mathcal{G}(P)) \cong \mathrm{Aut}(P)$. In particular, for the pair groupoid, bisections are in bijection with diffeomorphisms on the manifold, i.e. $\mathcal{B}(\mathrm{Pair}(M)) \cong \mathrm{Diff}(M)$, and local bisections are in bijection with diffeomorphisms defined locally on the manifold, i.e. $\mathcal{B}_{loc}(\mathrm{Pair}(M)) \cong \mathrm{Diff}_{loc}(M)$ via the correspondence $\sigma(x) = (\varphi_\sigma(x), x)$ which lies in $\mathrm{Pair}(M)$ for any x in $U \subset M$.

A smooth map $\phi : \mathcal{G} \longrightarrow \mathcal{G}'$ between two Lie groupoids over M, that is, a global map in the sense of Definition 1.2, induces a map

$$\mathcal{B}(\phi) : \mathcal{B}(\mathcal{G}) \longrightarrow \mathcal{B}(\mathcal{G}'),\ \sigma \mapsto \phi \circ \sigma. \tag{18}$$

In fact, since ϕ preserves the source and the target maps, we have $s' \circ (\phi \circ \sigma) = s \circ \sigma = \mathrm{Id}_M$ and the map $\varphi_{\phi\circ\sigma} = t' \circ (\phi \circ \sigma) = t \circ \sigma = \varphi_\sigma$ is a diffeomorphism on M. Moreover $\mathcal{B}(\phi)$ is a group morphism if ϕ is a groupoid morphism. In fact, for any σ, σ' in $\mathcal{B}(\mathcal{G})$ and for any x in M we have

$$\begin{aligned}\mathcal{B}(\phi)(\sigma' \bullet \sigma)(x) &= \phi(\sigma'(\varphi_\sigma(x)\sigma(x)) \\ &= \phi(\sigma'(\varphi_\sigma(x))\phi(\sigma(x)) = \big(\mathcal{B}(\phi)(\sigma') \bullet \mathcal{B}(\phi)(\sigma)\big)(x).\end{aligned}$$

If $\phi : \mathcal{G} \twoheadrightarrow \mathcal{G}'$ is a (smooth) *local* map between two Lie groupoids $\mathcal{G} \rightrightarrows M$, and $\mathcal{G}' \rightrightarrows M$, the composite map $\phi \circ \sigma$ is only defined for a local bisection σ of $\mathcal{G}$ taking values in the domain of ϕ. Assume $\phi : \mathcal{G} \twoheadrightarrow \mathcal{G}'$ is defined on a neighborhood $\mathcal{U}$ of the diagonal $u(M)$ in $\mathcal{G}$, where u is the unit on $\mathcal{G}$ and that $s' \circ \phi = s$, $t' \circ \phi = t$, where s, s' are the source maps on

$\mathcal{G}$ and $\mathcal{G}'$, t, t' the target maps on $\mathcal{G}$ and $\mathcal{G}'$. Let σ be a local bisection of $\mathcal{G}$ with support in an open set $U \subset M$ such that $\sigma(U) \subset \mathcal{U}$. Then $\sigma' := \phi \circ \sigma : U \to \mathcal{G}'$ defines a bisection of $\mathcal{G}'$. Indeed, we have $s' \circ \sigma' = s' \circ \phi \circ \sigma = s \circ \sigma = \mathrm{Id}_U$ and $t' \circ \sigma' = t' \circ \phi \circ \sigma = t \circ \sigma$ is a local diffeomorphism between U and an open subset of M. This way, ϕ induces a map

$$\mathcal{B}_{\mathrm{loc}}(\phi) : \mathcal{B}_{\mathrm{loc}}(\mathcal{G}) \longrightarrow \mathcal{B}_{\mathrm{loc}}(\mathcal{G}'),\ \sigma \mapsto \phi \circ \sigma, \tag{19}$$

between local bisections.

2. Direct connections on Lie groupoids

The first notion of a connection on a Lie groupoid, due to C. Ehresman in 1952 [Eh52], refers to a Lie algebroid connection on its Lie algebroid, that is, a bundle section of its anchor map, and is nowadays called an *infinitesimal connection*. A notion of *connection* specific to Lie groupoids was developed by A. Kock in the framework of Synthetic Differential Geometry in the 80's, to integrate infinitesimal representations of Lie groupoids [Koc89]. Such connections are required to preserve the units, to be invariant under inversion, and allow a notion of *curvature* which measures the obstruction to preserving general composition of arrows. A modern exposition of this approach can be found in [Koc17].

Later, *linear direct connections* were introduced by N. Teleman in 2004-2005 [Te04, Te07] specifically on the frame groupoid of a vector bundle $E \to M$, in order to extract the essence of the concept of parallel transport on E along geodesic curves in M suitable to describe the geometric content of the Chern classes of E. In the paper [KT06], which provides a comprehensive treatement of Teleman's approach, direct connections are only given for the frame groupoid of a vector bundle E. Teleman's coauthor J. Kubarski later proposed a generalisation to general Lie groupoids, still named linear direct connections, in the conference talk [Kub08]. Such direct connections are assumed to preserve the units, but are not necessarily invariant under inversion, and therefore provide a weaker version of Kock's connections, cf. [Koc07, §5].

In this section we present direct connections on groupoids and provide the details of some of the proofs in the litterature. In particular, we prove that a Lie groupoid which admits a direct connection is a gauge groupoid (Proposition 2.4), that a direct connection induces an infinitesimal connection on its Lie algebroid (Proposition 2.11), and conversely, if the base manifold has an affine connection, that a parallel transport on a principal bundle gives rise to a direct connection on its gauge groupoid (Proposition 2.13). Yet, not all direct connections are of this form (Example 2.18). We then recall the definition of the curvature of a direct connection and the known fact that, in the flat case, there is a one to one correspondence between direct connections and their infinitesimal connections, or, equivalently, parallel transports on the underlying principal bundle (Proposition 2.30).

2.1. **Direct connections.** For a manifold M, we call **diagonal domain** [2] in $M \times M$ any open neighborhood $\mathcal{U}_\Delta$ of the diagonal $\Delta := \{(x, x), x \in M\} \subset M \times M$.

Let M be endowed with a connection ∇ on TM. For v in TM, let c_v be the geodesic with initial data v. Let $\mathcal{D} = \{v \in TM \mid c_v \text{ is defined on } [0, 1]\}$. The exponential map of (M, ∇) is defined as

$$\exp : \mathcal{D} \to M \ , \ \exp(w) := c_w(1) \ .$$

For a point p in M we write $\mathcal{D}_p = \mathcal{D} \cap T_pM$ and $\exp_p(w) = \exp_{|\mathcal{D}_p}(w)$. The map exp gives a diffeomorphism between what we call a **diagonal exponential domain** $\mathcal{U}_\Delta$ in $M \times M$ and a neighbourhood of the zero section in TM.

Remark 2.1. A typical example is the Levi-Civita connection on a Riemannian manifold M with positive injectivity radius, and the any diagonal domain of the form

$$\mathcal{U}_{\Delta,r} = \{(y, x) \in M \times M,\ d(y, x) \leqslant r/2\}, \tag{20}$$

whose width can be adjusted within the range $0 < r < r_{inj}$.

[2] This terminology is borrowed from [DDD19, Def. 83]. They are called first neighbourhoods of the diagonal in [Koc17, §1].

Definition 2.2. A **direct connection** on a Lie groupoid $\mathcal{G} \rightrightarrows M$ is a smooth[3] local right inverse of the anchor map which preserves the units, that is, a local map $\Gamma : \mathrm{Pair}(M) \ast\!\!\longrightarrow \mathcal{G}$ over M, defined on a diagonal domain $\mathcal{U}_\Delta \subset M \times M$, such that

(1) $\Gamma(x, y)$ in $\mathcal{G}^x_y$ for all $(x, y) \in \mathcal{U}_\Delta$,
(2) $\Gamma(x, x) = 1_x \in \mathcal{G}^x_x$ for all $x \in M$.

We call a direct connection **global** if it is defined on the whole pair groupoid $\mathrm{Pair}(M)$.

Two connections are called **germ equivalent** if they are germ equivalent as local maps, i.e. if they agree on some common diagonal domain.

Example 2.3. Let $P = M \times G$ be the trivial principal G-bundle over a smooth manifold M. Let $g : M \to G$ be a smooth function, and $p_0 : x \mapsto (x, g(x))$ the corresponding global section of P. This gives rise to a (global) direct connection $\Gamma_g : \mathrm{Pair}(M) \to P \times_G P \simeq M \times G \times M$

$$\Gamma_g(y, x) = [(y, g(y)),\, (x, g(x))] = \left[(y, 1),\, (x, g(x)\, g(y)^{-1})\right].$$

Lie groupoids equipped with a direct connection are gauge groupoids if the base manifold is connected. We shall henceforth work under the assumption that the base manifold is connected.

Proposition 2.4. *A Lie groupoid $\mathcal{G} \rightrightarrows M$ over a connected manifold which can be equipped with a direct connection $\Gamma : \mathcal{P}(M) \ast\!\!\to \mathcal{G}$, is a gauge groupoid.*

Proof. By Theorem **??**, it is enough to show that $\mathcal{G}$ admits a section atlas. Fix a point $x_0 \in M$ and suppose that Γ is defined on a diagonal domain $\mathcal{U}_\Delta \subset M \times M$. Let $(U_\alpha)_{\alpha\in A}$ be an open cover of M such that $\bigcup_{\alpha\in A}(U_\alpha \times U_\alpha) \subset \mathcal{U}$. Then Γ is well defined on any pair of points laying in the same open set U_α.

We first show that for any $y \in M$ the fibre $\mathcal{G}^y_{x_0}$ is not empty. Since M is connected, one can choose a path $\gamma : [0, 1] \to M$ connecting x_0 and y. Then there exist finitely many indices $\alpha_1, \ldots \alpha_k$ such that the corresponding open sets U_{α_i} cover the image of the path. Order them in such a way that the consecutive intersections are not empty. For any $i = 1, ..., k$, choose a point $x_i \in U_{\alpha_i} \cap U_{\alpha_{i+1}}$. Then the composite arrow

$$\Gamma(y, x_k)\Gamma(x_k, x_{k-1}) \ldots \Gamma(x_2, x_1)\Gamma(x_1, x_0)$$

belongs to $\mathcal{G}^y_{x_0}$, and this shows the claim.

Now for every $\alpha \in A$ choose a point $x_\alpha \in U_\alpha$ and apply the result: the fibre $\mathcal{G}^{x_\alpha}_{x_0}$ is not empty and one can choose an arrow $\xi^{x_\alpha}_{x_0} \in \mathcal{G}^{x_\alpha}_{x_0}$. Finally, for any $x \in U_\alpha$, set

$$\sigma_\alpha(x) = \Gamma(x, x_\alpha)\xi^{x_\alpha}_{x_0} \in \mathcal{G}^x_{x_0}.$$

This gives a section atlas $(\sigma_\alpha : U_\alpha \to \mathcal{G}^{U_\alpha}_{x_0})_{\alpha\in A}$. ∎

Assuming that the base manifold M is connected is no restriction, since if this is not the case one can restrict to its connected components. Thus, from now on we consider gauge groupoids $\mathcal{G}(P) \rightrightarrows M$ associated to principal G-bundles $P \to M$ on a connected base manifold.

[3] Non necessarily smooth direct connections are considered in [Te04, Te07].

2.2. **Direct connections on frame groupoids.** In this paragraph, we specialise to a frame groupoid $\mathrm{Iso}(E) \rightrightarrows M$ of a (finite rank) vector bundle $\pi_E : E \to M$. It is a gauge groupoid $\mathcal{G}(FE) \rightrightarrows M$ with vertex bundle given by the frame bundle $\pi : FE \to M$.

We borrow from [DDD19, Definition 86] the notion of **parallelism** on a smooth vector bundle $E \to M$. It is a local smooth section

$$U : M \times M \ast\!\!\longrightarrow E^* \boxtimes E \tag{21}$$

of the external tensor product $E^* \boxtimes E \to M \times M$, which is defined on a diagonal neighborhood and such that $U(x,x) = \mathrm{Id}_x$ for any $x \in M$. We recall that the external tensor product $E_1 \boxtimes E_2 \to M_1 \times M_2$ of two vector bundles $\pi_i : E_i \to M_i$ is given by $E_1 \boxtimes E_2 := \mathrm{pr}_1^* E_1 \otimes \mathrm{pr}_2^* E_2$, where $\mathrm{pr}_i : M_1 \times M_2 \to M_i$ is the canonical projection.

Clearly a direct connection $\Gamma : \mathrm{Pair}(M) \ast\!\!\longrightarrow \mathrm{Iso}(E)$ on the frame groupoid yields a parallelism. Here is an example taken from [DDD19, Example 88].

Example 2.5. Let r be the rank of E and $(U_\alpha, \theta_\alpha)$ be a trivialising system for E, with $\theta_\alpha : E_{|U_\alpha} = \pi_E^{-1}(U_\alpha) \xrightarrow{\cong} U_\alpha \times \mathbb{R}^r$. Let $\psi_\alpha : F(E)_{|U_\alpha} = \pi^{-1}(U_\alpha) \xrightarrow{\cong} U_\alpha \times GL_r(\mathbb{R})$ be the corresponding trivialisation of $F(E)$, given by the map $p \mapsto \psi_\alpha(p) := \theta_\alpha \circ p$, where a frame $p \in F(E)_x$ above a point $x \in U_\alpha$ is seen as an isomorphism $\mathbb{R}^r \to E_x$ and θ_α restricts to an isomorphism $E_x \to \{x\} \times \mathbb{R}^r$. Then, the map $\Gamma_\alpha : U_\alpha \times U_\alpha \to \mathrm{Iso}(E)_{|U_\alpha}$ defined by

$$\Gamma_\alpha(y,x) := \psi_\alpha^{-1}(x, \mathrm{Id}) \circ (\psi_\alpha^{-1}(y, \mathrm{Id}))^{-1}, \qquad \text{for } x, y \in U_\alpha,$$

is a direct connection on $\mathrm{Iso}(E)$ defined on the diagonal domain $\mathcal{U}_\Delta = \bigcup_\alpha U_\alpha \times U_\alpha$.

□

Conversely, a parallelism on a vector bundle yields a direct connection on its frame groupoid.

Proposition 2.6. *A parallelism $U : M \times M \ast\!\!\longrightarrow E^* \boxtimes E$ defines a direct connection*

$$\begin{aligned} \Gamma : \mathrm{Pair}(M) &\ast\!\!\longrightarrow \mathrm{Iso}(E) \\ (y,x) &\mapsto U(x,y). \end{aligned}$$

Proof. The parallelism sends a pair (x,y) in $M \times M$ to $U(x,y)$ in $E_x^* \times E_y$ and all we need to prove is the invertibility of the maps $U(x,y) : E_x \longrightarrow E_y$ for a pair (x,y) in some local neighborhood of the diagonal. Let us consider a point x_0 in M and a trivialising neighborhood U_{x_0} of x_0 for E, so that $E|_{U_{x_0}} \simeq U_{x_0} \times \mathbb{R}^r$ where r is the rank of E. The parallelism induces a map $\Gamma(\cdot, x) : U_{x_0} \longrightarrow \mathrm{End}(\mathbb{R}^r)$ which sends an element y in U_{x_0} to $\Gamma(y, x_0)$ in $E_{x_0}^* \times E_y \simeq \mathrm{End}(\mathbb{R}^n)$. Since $\Gamma(x_0, x_0) = \mathrm{Id}_{x_0}$, it sends x_0 to $\mathrm{Id}_{\mathbb{R}^r}$ which is invertible in $\mathrm{End}(\mathbb{R}^n)$. The local inverse theorem then yields the existence of a local neighborhood $V_{x_0} \subset U_{x_0}$ such that the restriction $\Gamma(\cdot, x_0)|_{V_{x_0}}$ is invertible, i.e $\Gamma(\cdot, x_0)|_{V_{x_0}} : V_{x_0} \longrightarrow \mathrm{Iso}(\mathbb{R}^n)$. The parallelism U therefore maps an element (x,y) of the diagonal neighborhood $\bigcup_{x_0 \in M} V_{x_0} \times V_{x_0}$ to $U(x,y)$in $\mathrm{Iso}(E_x, E_y)$ which shows that $\Gamma(y,x) := U(x,y)$ defines a direct connection on $\mathrm{Iso}(E)$.

□

2.3. **Infnitesimal connections on Lie algebroids.** The terminology for connections on Lie algebroids is motivated by that of principal (or Ehresman) connections on principal bundles.

Definition 2.7. [MK05, Definition 5.2.5] A **connection on a Lie algebroid** $\mathcal{L} \to M$ is a splitting of the anchor in the exact sequence of vector bundles:

$$0 \longrightarrow \mathrm{Ker}(a) \overset{\iota}{\hookrightarrow} \mathcal{L} \overset{a}{\longrightarrow} TM \longrightarrow 0, \tag{22}$$

i.e., a vector bundle map $\delta : TM \to \mathcal{L}$ such that $a \circ \delta = \mathrm{Id}_{TM}$.

It yields an isomorphism of vector bundles

$$\begin{array}{rcl} TM \times \mathrm{Ker(a)} & \overset{\simeq}{\longrightarrow} & \mathcal{L} \\ (X, k) & \longmapsto & \delta(X) + \iota(k). \end{array} \tag{23}$$

Example 2.8. [MK05, §5.3] If $\mathcal{L} = A(P) = TP/G$ for some principal G-bundle $P \to M$, the exact sequence (8) yields the Atiyah exact sequence

$$0 \longrightarrow P \times_G \mathfrak{g} \overset{\iota}{\hookrightarrow} A(P) \overset{a}{\longrightarrow} TM \longrightarrow 0 \tag{24}$$

since the anchor map on $A(P)$ coincides with the differential $T\pi|_M : TP \to TM$ of the canonical projection $\pi : P \to M$ and a connection on the Lie algebroid $A(P)$ is an infinitesimal connection on P.

Example 2.9. We now specialise to the frame bundle $P = FE$ and the Lie algebroid of **derivations** $\mathcal{L} := \mathrm{Der}(E)$ (see Example 1.11 (5)) whose anchor is given by $a : D \mapsto X_D$ with kernel $\mathrm{End}(E)$. Consequently, there is an exact sequence of vector bundles (see eq. (25))

$$0 \longrightarrow \mathrm{Ker}(a) = \mathrm{End}(E) \longrightarrow \mathrm{Der}(E) \overset{a}{\longrightarrow} TM \longrightarrow 0 \tag{25}$$

which yields an isomorphism of vector bundles

$$\begin{array}{rcl} TM \times \mathrm{End}(E) & \overset{\simeq}{\longrightarrow} & \mathrm{Der}(E) \\ (X, L) & \longmapsto & \delta(X) + L. \end{array} \tag{26}$$

2.4. **Infinitesimal connections induced by direct connections.**

Definition 2.10. If $\mathcal{G} \rightrightarrows M$ is a Lie groupoid, we call **infinitesimal connection on** $\mathcal{G}$, a connection on its Lie algebroid $\mathcal{L}(\mathcal{G})$.

A Lie algebroid admitting a connection is necessarily transitive (the anchor is surjective) and therefore it is the Atiyah algebroid $A(P) = TP/G$ of a principal G-bundle $P \to M$. Conversely, any transitive Lie algebroid admits a connection, cf. also [MK05, Corollary 5.2.7]. Consequently, we henceforth consider Atiyah algebroids $\mathcal{L} = A(P)$.

Proposition 2.11. *Let $\mathcal{G}(P) \rightrightarrows M$ be a gauge groupoid endowed with a direct connection $\Gamma : \mathrm{Pair}(M) \ast\!\!\longrightarrow \mathcal{G}(P)$. Then the differential of Γ along the diagonal defines an infinitesimal connection*

$$\delta^\Gamma = D\Gamma_{|M} : TM \to A(P)$$

$$\dot{c_x}(0) \mapsto \delta^\Gamma(\dot{c_x}(0)) = \frac{d}{dt}\Gamma(c_x(t), x)_{|t=0}, \tag{27}$$

where $c_x : [0, 1] \to M$ is a smooth curve with initial point $x = c_x(0)$.

Proof. Applying Lemma 1.12 to $\mathcal{G}_1 = \mathrm{Pair}(M)$, $\mathcal{G}_2 = \mathcal{G}$, and $\phi = \Gamma$, we build the infinitesimal connection from the differential of Γ

$$\delta^\Gamma := D\Gamma_{|M} : TM \longrightarrow \mathcal{L}\mathcal{G}.$$

The differential of Γ along the diagonal is then computed by fixing a source point $x \in M$ and differentiating Γ with respect to the target variable at x. Since $\mathcal{L}(\mathrm{Pair}(M)) = TM$, the differential $D\Gamma_{|M}$ at x is indeed defined on the tangent space T_xM. The vector $\frac{d}{ds}\Gamma(c_x(s), x)_{|s=0}$ in eq. (27) belongs to the tangent space of the source fibre $\mathcal{G}(P)_x$ at the point $\Gamma(x, x) = 1_x$. Consequently, the map δ^Γ takes values in the Lie algebroid $\mathcal{L}(\mathcal{G}(P)) \cong A(P)$. Finally, δ^Γ yields a splitting of the anchor $a = dt_{|T_{1_x}\mathcal{G}_x} = d\pi : A(P) \to TM$, which brings $\frac{d}{ds}\Gamma(c_x(s), x)\big|_{s=0}$ back to the derivative of the target of $\Gamma(c_x(s), x)$ at $s = 0$, which is precisely $\dot{c}_x(0)$. □

This statement also provides an independent proof of the fact that a Lie groupoid on a connected manifold equipped with a direct connection is a gauge groupoid, cf. Proposition 2.4. Indeed, since a direct connection on a $\mathcal{G}$ induces a connection on $\mathcal{L}(\mathcal{G})$, the latter is transitive and hence an Atiyah algebroid $A(P)$. If the base manifold M is connected, this Lie algebroid then necessarily integrates to the gauge groupoid $\mathcal{G}(P)$ [MK05, Corollary 3.5.18]. Hence, $\mathcal{G} = \mathcal{G}(P)$ is a gauge groupoid.

We now specialise to the frame groupoid $\mathrm{Iso}(E) = \mathcal{G}(FE)$ of a vector bundle $E \to M$.

Proposition 2.12. *Let $E \to M$ be a vector bundle whose frame groupoid $\mathrm{Iso}(E)$ is equipped with a direct connection Γ. Let $X \in T_xM$ and let $c_x : [0,1] \to M$ be a smooth curve in M starting at $c_x(0) = x$ and set $\dot{c}_x(0) = X \in T_xM$. The expression*

$$\nabla^\Gamma_X(f) := \frac{d}{dt}\left(\Gamma^{-1}(x, c_x(t))\, f(c_x(t))\right)\big|_{t=0}, \tag{28}$$

defined for any local smooth section f of E on a neighborhood U of x, gives rise to a linear derivation $f \mapsto \nabla^\Gamma_X f$ on E. Together with the infinitesimal connection

$$\delta^\Gamma(X)(f) = \frac{d}{dt}\left(\Gamma(c_x(t), x)\, f(x)\right)\big|_{t=0} \in E_x$$

it yields a decomposition of the tangent map $D_xf : T_xM \to T_{f(x)}E$ to f at any point x in M:

$$D_xf(X) = \delta^\Gamma(X)(f) + \nabla^\Gamma_X f \tag{29}$$

into a vertical part $\nabla^\Gamma_X f$ and a horizontal part $\delta^\Gamma(X)f$.

Equivalently, we have

$$X\langle\phi, f\rangle = \delta(X)(\ell_\phi)f + \langle D_{\delta(X)}(f), \phi\rangle \quad \forall \phi \in E^*_x. \tag{30}$$

Proof. The map

$$\nabla^\Gamma_X : f \longmapsto \frac{d}{dt}\left(\Gamma^{-1}(x, c_x(t))\, f(c_x(t))\right)\big|_{t=0}$$

clearly defines a linear derivation on E. From the fact that $\Gamma(c_x(t), x)\,\Gamma^{-1}(c_x(t), x) = \mathrm{Id}_{c_x(t)}$ for all times t it follows that

$$\begin{aligned}
D_xf(X) &= \frac{d}{dt} f(c_x(t))\big|_{t=0} \\
&= \frac{d}{dt}\left(\Gamma(c_x(t), x)\,\Gamma^{-1}(x, c_x(t))\, f(c_x(t))\right)\big|_{t=0} \\
&= \frac{d}{dt}\left(\Gamma(c_x(t), x)\, f(x)\right)\big|_{t=0} + \frac{d}{dt}\left(\Gamma^{-1}(x, c_x(t))\, f(c_x(t))\right)\big|_{t=0} \\
&= \delta^\Gamma(X)f + \nabla^\Gamma_X f.
\end{aligned}$$

Note that $\xi = \delta^\Gamma(X)$ lies in the Atiyah algebroid $A(FE) = T^{\text{lin}}E$ of the frame bundle FE. Proposition 5.3.5 in [MK05] applied to the frame bundle then gives the following identification

$$\nabla^\Gamma_X(f) = D_{\delta^\Gamma(X)}(f) = \rho(\delta^\Gamma X)(f),$$

with $\rho : A(FE) \longrightarrow \text{Der}(E)$ as in eq. (14). Inserting this in eq. (15) gives eq. (30). □

2.5. **Direct connections defined by parallel transport.** We now explore the relation between parallel transport and direct connections.

Proposition 2.13. *Given a smooth manifold M endowed with an affine connection and the gauge groupoid $\mathcal{G}(P) \rightrightarrows M$ of a principal G-bundle $\pi : P \to M$ endowed with a principal connection, the parallel transport τ on P along small geodesics on M defines a direct connection Γ^τ on $\mathcal{G}(P)$.*

Proof. The parallel transport $\tau_c(y,x)$ along any curve $[0,1] \ni t \to c(t) \in M$ joining x to y can be seen as an element of the subset $P_y \times_G P_x$ of the gauge groupoid $\mathcal{G}(P) = P \times_G P$, by means of the identification

$$\tau_c(y,x) \longmapsto \Gamma^{\tau_c}(y,x) = [\tau_c(y,x)(p_0), p_0] \quad \text{for any choice of } p_0 \in P_x.$$

This is well defined in $P_y \times_G P_x$ since for any other element $p_0' \in P_x$, there exists a unique element g in G such that $p_0' = p_0 g$. The equality $[\tau_c(y,x)(p_0'), p_0'] = [\tau_c(y,x)(p_0), p_0]$ follows from the G-equivariance of $\tau_c(y,x)$.

Since M is endowed with an affine connection pairs of points in the exponential diagonal domain are linked by a unique geodesic. When the curve c is the geodesic c_x^y joining x and y we shall set $\tau(y,x) := \tau_c(y,x)$. This notational convention will apply to any parallel transport along geodesics linking any pair of points (x,y) in an exponential diagonal domain $\mathcal{U}_\Delta$. The parallel transport $\tau(y,x)$ along the unique geodesic linking x and y defines a direct connection $(x,y) \longmapsto \Gamma^\tau(y,x)$. □

Here is a first trivial example.

Example 2.14. On the trivial Lie groupoid $\mathcal{G}(P) = M \times G \times M$, for $P = M \times G$, the horizontal distribution $H_{(x,g)}P = T_xM$ gives the parallel transport $\tau_c(y,x)(x,g) = (y,g)$ along any curve c linking x to y, and therefore the direct connection $\Gamma(y,x) = [(y,1_G),(x,1_G)]$ of Example 2.3 with $G = \{1\}$. □

Example 2.15. With the notations of Example 2.3, any element $p \in P$ can be written $p = (x, g(x)\,h)$ for a unique h in G and the $\mathfrak{g}$-valued map defined on P by $\omega(x, g(x)\,h) := g(x)^{-1}\,dg(x)$ yields a principal connection on P. The direct connection Γ_g is induced by the corresponding parallel transport.

In Proposition 2.11, we saw that a direct connection Γ on $\mathcal{G}(P)$ induces an infinitesimal connection $\delta^\Gamma : TM \to A(P)$ and hence a horizontal distribution in TP. We now show that if Γ is the direct connection defined by a parallel transport on P, as in Proposition 2.13, then δ^Γ coincides with the infinitesimal connection induced by the parallel transport.

Theorem 2.16. *Let $P \to M$ be a principal G-bundle with an infinitesimal connection $\delta : TM \to A(P)$ and consider the direct connection Γ^δ on $\mathcal{G}(P)$ defined by the parallel* *transport along small geodesics induced by δ. Then, δ coincides with the associated infinitesimal connection*

$$\delta(\dot{c_x}(0)) = \frac{d}{dt}\Gamma^\delta(c_x(t), x)\big|_{t=0}. \tag{31}$$

Here, as before $c_x : [0,1] \to M$ is a smooth curve with initial point $c_x(0) = x$.

Proof. Given an infinitesimal connection $\delta : TM \to A(P)$ and its parallel transport τ_c along small geodesics c, consider the associated direct connection Γ^δ : Pair$(M) \ast\!\!\longrightarrow \mathcal{G}(P)$ given by $\Gamma^\delta(y,x) = [\tau_c(y,x)(p), p]$ for any choice of $p \in P_x$. With some abuse of notation, we denote by $\tau(c_x(t),x)(p)$ the parallel transport of $p \in P_x$ to the fibre above $c_x(t)$ along the curve c_x with initial point $c_x(0) = x$.

The infinitesimal connection $\delta^{\Gamma^\delta} : TM \to A(P)$ of the direct connection Γ^δ reads

$$\delta^{\Gamma^\delta}(\dot{c_x}(0)) = \frac{d}{dt}\Gamma^\delta(c_x(t), x)\big|_{t=0} = \frac{d}{dt}[\tau(c_x(t), x)(p), p]\big|_{t=0}.$$

By uniqueness of the horizontal lift, the tangent vector at $t = 0$ to the curve $\gamma(t) = \tau(c_x(t), x)(p)$ gives the value of $\delta(\dot{c_x}(0))$. Taking into account the action of G on TP, which makes the choice of p irrelevant, one can identify $\delta^{\Gamma^\delta}(\dot{c_x}(0))$ to the class of $\dot\gamma(0)$ in TP/G from which it follows that

$$\delta^{\Gamma^\delta}(\dot{c_x}(0)) = \chi_P(\dot\gamma(0)) = \delta(\dot{c_x}(0)),$$

for any $\dot{c_x}(0) \in T_xM$. □

The above proposition shows how infinitesimal connections can be integrated to direct connections. However, as we shall see in the next section the correspondence between direct connection and infinitesimal connection is not one-to-one. This constrasts with *path connections* on groupoids, cf. [MK05] that integrate infinitesimal connections along paths in the base manifold. If the base manifold is connected, path connections are proven to be in one to one corespondence with infinitesimal connections [MK05, Theorem 6.3.5].

We now specialise to the frame groupoid $FE \to M$ of a vector bundle $E \to M$ equipped with a linear connection ∇. The associated **parallel displacement on** E (or **parallel transport**) along a curve $[0,1] \ni t \to c(t) \in M$ from the point $x = c(0) \in M$ to the point $y = c(1) \in M$, is the map $\tau_c(y,x) : E_x \to E_y,\ e_0 \mapsto e(1)$ where $[0,1] \ni t \mapsto e(t) \in E$ solves the equation $\nabla_{\dot c(t)} e = 0$ for any $t \in [0,1]$ with $e(0) = e_0$.

As before, when the curve c is the geodesic c_x^y which links x to y we shall simply write

$$\tau(y,x) : E_x \longrightarrow E_y. \tag{32}$$

This notational convention will apply to any parallel transport along geodesics linking two specified points.

We have $\nabla_X(f) = \frac{d}{dt}\left(\tau^{-1}(c_x(t), x)\, f(c_x(t))\right)\big|_{t=0}$.

The parallel transport on vector bundles is considered by Teleman in [Te04, Te07], and appears in [DDD19] under the name *parallelism*, see eq. (21).

Corollary 2.17. [KT06, Remark 2] *Let M be a smooth Riemannian manifold endowed with a connection with positive injectivity radius and let $E \to M$ be a vector bundle equipped* *with a linear connection. The induced parallel transport τ on E along small geodesics of M defines a direct connection:*

$$\Gamma^{\tau} : \mathcal{U}_{\Delta} \longrightarrow \mathrm{Iso}(E), (y, x) \mapsto \tau(y, x). \tag{33}$$

on the frame groupoid $\mathrm{Iso}(E) \rightrightarrows M$*.*

Proof. Let $[0,1] \ni t \to c(t)$ in M be a curve joining x to y in M. The fact that the parallel transport $\tau_c(y, x) : E_x \to E_y$ is a linear isomorphism of the fibres follows from the vector space structure of V and from the fact that it is invertible (with inverse given by the horizontal lift $e^{-1}(t)$ along the inverse curve $c^{-1}(t) = c(1-t)$). Therefore $\tau_c(y, x)$ in $\mathrm{Iso}(E)$ for any curve c and any two points (y, x) for which it is defined. On a manifold M endowed with an affine connection, for any two points x and y in an exponential diagonal domain $\mathcal{U}_{\Delta}$, letting c be the unique geodesic linking them yields the map $\tau(y, x)$ in $\mathrm{Iso}(E)_x^y$ (as before we drop the mention of the geodesic) which in turn gives rise to the direct connection (33). □

We borrow from [KT06] an easy and illustrative example.

Example 2.18. [KT06, Example 5] Let $M = \mathbb{R}$ with points x, global vector field $\partial_x = \frac{d}{dx}$ on M, flat linear connection $\nabla^M_{\partial_x}(f(x)\,\partial_x) = f'(x)\,\partial_x$ and geodesics given by the segments parametrized by x.

Let $E = M \times \mathbb{R}$ be the trivial bundle on $M = \mathbb{R}$, with global section $e_1 : M \to E$, $x \mapsto e_1(x) = (x, 1)$ in E_x. A linear connection $\nabla : \Gamma(TM) \otimes \Gamma(E) \to \Gamma(E)$ on E (necessarily flat) is given by its Christoffel symbol $k \in C^\infty(M)$ such that $\nabla_{\partial_x} e_1 = k\, e_1$. The induced parallel transport of a vector $\xi_0\, e_1(x) \in E_x$ along a geodesic from x to y is the isomorphism of vector space $\tau(y, x) : E_x \to E_y$ which assigns to a vector $\xi_0\, e_1(x) \in E_x$ the vector $\xi(y) = e^{K(y)-K(x)}\xi_0 \in E_y$, where $K(x) = \int -k(x)\, dx$. The direct connection on $\mathrm{Iso}(E)$ induced by ∇ is the global map $\Gamma^{\nabla} : \mathrm{Pair}(M) \to \mathrm{Iso}(E)$ given by

$$\Gamma^{\nabla}(y, x) : E_x \to E_y,\ \ e_1(x) \mapsto \tau(y, x)\, e_1(x) = e^{K(y)-K(x)}\, e_1(y). \tag{34}$$

In contrast, the two smooth maps

- $\alpha(y, x)\, e_1(x) = e^{(y-x)+(y-x)^2}\, e_1(y)$,
- $\beta(y, x)\, e_1(x) = e^{(y-x)+(y-x)^3}\, e_1(y)$,

define linear isomorphisms $E_x \to E_y$ such that $\alpha(x, x) = \mathrm{Id}_{|E_x}$ and $\beta(x, x) = \mathrm{Id}_{|E_x}$ which yield (global) direct connections on $\mathrm{Iso}(E)$. But they are not parallel transports, since they are not of the form eq. (34). □

2.6. **Curvature of direct connections.** Given two (small) geodesics α and β on M, from x to y and from y to z respectively, the compositon $\beta \circ \alpha$ is not necessarily the geodesic from x to z. The parallel transport τ_c along geodesics defined by a principal connection on a principal bundle $P \to M$, then, does not necessarily satisfy the identity $\tau_{c_2}(z, y) \circ \tau_{c_1}(y, x) = \tau_{c_3}(z, x)$. For the direct connection Γ^{τ} on the groupoid $\mathcal{G}(P)$ induced by the parallel transport as in eq. (33), this identity amounts to Γ^{τ} being a morphism of groupoids. In this section we introduce a *curvature* for a direct connection, which measures the obstruction to it being a groupoid morphism. Our definition slightly differs from that given by N. Teleman and J. Kubarski in [Te04, Te07, KT06, Kub08], and by A. Kock in [Koc07, Koc17], but it is equivalent.

Definition 2.19. Let $\Gamma : \mathrm{Pair}(M) \ast\!\!\longrightarrow \mathcal{G}$ be a direct connection defined on a diagonal domain $\mathcal{U}_\Delta$. For any $x \in M$, we set

$$\mathcal{U}^1_\Delta(x) = \{y \in M \mid (x,y),(y,x) \in \mathcal{U}_\Delta\} \subset M$$

and $I^\Gamma(\_,x) : \mathcal{U}^1_\Delta(x) \longrightarrow \mathcal{G}^x_x,\ y \mapsto I^\Gamma(y,x) := \Gamma(x,y)\,\Gamma(y,x)$. We call Γ **natural** if $I^\Gamma(y,x) = 1_x$ for all $x \in M$ and for all $y \in \mathcal{U}^1_\Delta(x)$. Similarly, for any $x \in M$, we set

$$\mathcal{U}^2_\Delta(x) = \{(z,y) \in M \times M \mid (y,x),(z,y),(z,x) \in \mathcal{U}_\Delta\} \subset M \times M$$

and call **curvature of** Γ **at** x the map $R^\Gamma(\_,\_,x) : \mathcal{U}^2_\Delta(x) \longrightarrow \mathcal{G}^x_x$ given by

$$R^\Gamma(z,y,x) := \Gamma(z,x)^{-1}\,\Gamma(z,y)\Gamma(y,x) \in \mathcal{G}^x_x, \qquad (z,y) \in \mathcal{U}^2_\Delta(x). \tag{35}$$

The direct connection Γ is **flat** if $R^\Gamma(z,y,x) = 1_x$ for any $x \in M$ and for any $(z,y) \in \mathcal{U}^2_\Delta(x)$. This is equivalent to the condition $\Gamma(z,y)\,\Gamma(y,x) = \Gamma(z,x)$ for any $x \in M$ and any $(z,y) \in \mathcal{U}^2_\Delta(x)$, or, equivalently, that $\Gamma : \mathrm{Pair}(M) \circ\!\!\rightarrow \mathcal{G}$ is a local groupoid morphism.

Remark 2.20. A flat direct connection Γ satisfies

$$1_x = \Gamma(x,y)\,\Gamma(y,x) \quad \forall x \in M, \forall y \in \mathcal{U}^1_\Delta(x), \tag{36}$$

as a consequence of the fact that $\Gamma(x,x) = 1_x$. Indeed, if eq. (35) holds, then for any $x \in M$ and any $y \in \mathcal{U}^1_\Delta(x)$ we have $(x,y) \in \mathcal{U}^2_\Delta(x)$ and

$$1_x = \Gamma(x,x)^{-1}\Gamma(x,y)\Gamma(y,x) = \Gamma(x,y)\Gamma(y,x).$$

Example 2.21. A direct connection Γ^τ given by a parallel transport as in eq. (34) of Example 2.18 satisfies condition eq. (36). The direct connections α and β in Example 2.18 have non trivial curvature, yet whereas β satifies eq. (36), the direct connection α does not. □

Example 2.22. Note that a direct connection of the form

$$\Gamma(y,x) = \sigma(y)\,\sigma(x)^{-1}, \tag{37}$$

for some smooth section $\sigma : U \mapsto \mathcal{G}^x$ and some given point $x \in M$, defines a flat connection.

Given a principal bundle $P \to M$ with structure group G with Lie algebra $\mathfrak{g}$, the **curvature form** of an infinitesimal connection $\delta : TM \to A(P)$ is the two form Ω^δ in $\Omega^2(M,\mathfrak{g})$ defined by $\Omega^\delta(X_1,X_2) := [\delta(\tilde{X}_1),\delta(\tilde{X}_2)]_{A(P)} - \delta\left([\tilde{X}_1,\tilde{X}_2]_{\mathfrak{X}(M)}\right)$ for any $X_i \in T_xM, i = 1,2$ and any vector field extension $\tilde{X}_i$ of X_i. So the flatness of the infinitesimal connection δ amounts to it being a morphism of Lie algebroids.

Remark 2.23. For the relation between the curvature R^Γ of a direct connection Γ on a groupoid $\mathcal{G}$ and the curvature $\Omega^{\nabla\delta}$ of the corresponding infinitesimal connection $\nabla^\Gamma : TM \to \mathcal{L}(\mathcal{G})$, we refer to [KT06, Lemma 11] for frame groupoids and [KT06, Theorem 12] for general locally trivial Lie groupoids.

2.7. **Flat direct connections.** A direct connection $\Gamma : \mathrm{Pair}(M) \ast\!\!\rightarrow \mathcal{G}$ is flat if and only if it is a local groupoid morphism. Such maps have been extensively studied by K. Mackenzie in [MK05, Chapter 6], in particular for what concerns the relationship with their infinitesimal Lie algebroid morphisms. We collect here some relevant results, the first of which is a description of the structure of a Lie groupoid which admits a flat direct connection.

Definition 2.24. A Lie groupoid $\mathcal{G} \rightrightarrows M$ is called **flat** if it is locally trivial and it admits an atlas of local decomposition maps $\sigma_\alpha : U_\alpha \to \mathcal{G}_x^{U_\alpha}$ (cf. Definition 1.4) whose transition functions $g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathcal{G}_x^x$ given by $g_{\alpha\beta}(x) = \sigma_\alpha(x)^{-1}\,\sigma_\beta(x)$ are locally constant, i.e. constant on each connected component.

In essence, the subsequent statement is a reformulation of [MK05, Example 6.1.7] in terms of direct connections. We call a Lie groupoid **flat** if it admits a section atlas (see Definition 1.4) with constant transition functions in which case we call the groupoid flat.

Proposition 2.25. *A Lie groupoid on a connected base admits a flat direct connection if and only if it is flat. In that case, it is locally exact, namely there is an open covering* $\{U_\alpha\}$ *of* M *and a section atlas* $\sigma_\alpha : U_\alpha \to \mathcal{G}_{U_\alpha}^x$ *with constant transition functions such that the flat connection* Γ *reads*

$$\Gamma|_{U_\alpha \times U_\alpha}(y, x) = \sigma_\alpha(y)\,\sigma_\alpha^{-1}(x). \tag{38}$$

Proof. ($\Rightarrow$) Assume there exists a local morphism $\Gamma\colon \mathcal{U}_\Delta \circ\!\!\longrightarrow \mathcal{G}$ defined on a diagonal domain $\mathcal{U}_\Delta$, in which case $\mathcal{G}$ is locally trivial by Proposition 2.4. We choose an open covering $\{U_\alpha\}$ of M such that $\bigcup_\alpha (U_\alpha \times U_\alpha) \subset \mathcal{U}_\Delta$ andt $x \in M$. For every α, fix a point $x_\alpha \in U_\alpha$ together with an arrow $\xi_\alpha \in \mathcal{G}_x^{x_\alpha}$ (which exists by the local triviality of $\mathcal{G}$). Then the collection of $\sigma_\alpha(x) := \Gamma(x, x_\alpha)\,\xi_\alpha$ builds a section atlas with constant transition functions. Moreover, for fixed α, the map

$$\begin{array}{ccc} \mathcal{G}_{U_\alpha}^{U_\alpha} & \longrightarrow & U_\alpha \times \mathcal{G}_x^x \times U_\alpha \\ \gamma & \longmapsto & (t(\gamma), \sigma_\alpha(t(\gamma))^{-1}\gamma\sigma_\alpha(s(\gamma)), s(\gamma)) \end{array}$$

is an isomorphism under which, for all $x, y \in U_\alpha$, we have

$$\Gamma(y, x) \equiv (y, 1_x, x). \tag{39}$$

($\Leftarrow$) Conversely, let $\sigma_\alpha\colon U_\alpha \to \mathcal{G}_{U_\alpha}^x$ be a section atlas with constant transition functions $g_{\alpha\beta}(x) = \sigma_\beta(x)^{-1}\,\sigma_\alpha(x)$ on the intersection $U_\alpha \cap U_\beta$, and let us set $\mathcal{U}_\Delta := \bigcup_\alpha (U_\alpha \times U_\alpha)$. The map $\Gamma : \mathcal{U}_\Delta \to \mathcal{G}$ given by eq. (38) is well defined. Indeed, the transition maps $g_{\alpha\beta}$ being constant on the intersection $U_{\alpha\beta} := U_\alpha \cap U_\beta$, for $(x, y) \in U_{\alpha\beta}^2$ we have

$$\begin{aligned} \Gamma|_{U_\alpha \times U_\alpha}(y, x)\ \left(\Gamma|_{U_\beta \times U_\beta}(y, x)\right)^{-1} &= \sigma_\alpha(y)\,\sigma_\alpha^{-1}(x)\,\sigma_\beta(x)\,\sigma_\beta^{-1}(y) \\ &= \sigma_\beta(y)\,\sigma_\beta^{-1}(y)\,\sigma_\alpha(y)\,g_{\beta\,\alpha}(x)\,\sigma_\beta^{-1}(x) \\ &= \sigma_\beta(y)\,g_{\alpha,\beta}(y)\,g_{\beta\,\alpha}(x)\,\sigma_\beta^{-1}(x). \end{aligned}$$

That $\Gamma|_{U_\alpha \times U_\alpha}$ defines a morphism is easily verified.

Example 2.26. With the notations of Example 2.3, the Lie groupoid $\mathcal{G}(P) = M \times G \times M$ is flat and the direct connection $\Gamma_g(x, y) = [(y, 1), (x, g(x)\,g(y)^{-1})]$ is of the form (37) and hence flat, which we can also see directly since

$$\Gamma_g(z, y)\,\Gamma_g(y, x) = \left[(z, 1), (x, g(x)\,g(z)^{-1})\right] = \Gamma_g(z, x).$$

Example 2.27. The fundamental groupoid $\Pi(M)$ on a connected manifold M has constant transition maps. Hence, by Proposition 2.25, it admits a flat direct connection $\Gamma_0 : \mathrm{Pair}(M) \circ\!\!\longrightarrow \Pi(M)$. It can be constructed as follows. Let $\{U_\alpha\}$ be a cover of M by simply connected open subsets, then $\mathcal{U}_\Delta = \bigcup_\alpha (U_\alpha \times U_\alpha)$ defines a diagonal domain. For any $(x, y) \in \mathcal{U}_\Delta$, we define $\Gamma_0(x, y)$ as the homotopy class of any path in $\mathcal{U}_\Delta$ connecting x and y. This is well defined as any two paths in U_α are homotopic. □

The importance of the direct connection Γ_0 lies in a factorisation result shown in MK, which we reformulate here in the language of direct connections.

Proposition 2.28. [MK05, Prop. 6.1.8] *Let M be connected.*

(1) The flat direct connection $\Gamma_0 : \mathrm{Pair}(M) \circ\!\!\longrightarrow \Pi(M)$ is uniquely determined up to germ equivalence.

(2) Any flat direct connection $\Gamma : \mathrm{Pair}(M) \circ\!\!\longrightarrow \mathcal{G}$ on a locally trivial Lie groupoid $\mathcal{G}$ on M factorises in a unique way through Γ_0, i.e. there exists a unique groupoid morphism

$$H : \Pi(M) \to \mathcal{G}, \tag{40}$$

such that $\Gamma = H \circ \Gamma_0$.

Remark 2.29. Clearly, the first assertion follows from the second one applied to $\mathcal{G} = \Pi(M)$.

There is a one-to-one correspondence between flat infinitesimal connections and flat direct connections, a known fact [MK05, Corollary 6.2.7] that we briefly spell out for the sake of completeness.

Proposition 2.30. *Let $\mathcal{G} \rightrightarrows M$ be a locally trivial Lie groupoid.*

(1) The infinitesimal connection δ^Γ defined by a flat direct connection Γ on $\mathcal{G}$ is flat.

(2) Conversely, a flat infinitesimal connection $\delta : TM \to \mathcal{L}(\mathcal{G})$ integrates to a flat direct connection Γ^δ on $\mathcal{G}$, which is unique up to germ equivalence.

In particular, a direct connection is flat if and only if its infinitesimal connection is flat.

Proof. The first assertion follows from Lemma 1.12 applied to the local map $\Gamma : \mathrm{Pair}(M) \twoheadrightarrow \mathcal{G}$ since $D\Gamma_M = \delta^\Gamma$. The flateness of the infinitesimal connection induced by a flat direct connection follows from the Lie algebroid morphism property of the tangent map along the diagonal to a local groupoid morphism [MK05, §3.5]. The second assertion follows from part (2) of Proposition 1.13, including the uniqueness up to germ equivalence. □

Example 2.31. [MK05, Example 6.1.7] We have $\mathcal{L}(\Pi(M)) = \mathcal{L}(\mathcal{P}(M)) = TM$, so the identity map $\mathrm{Id} : TM \longrightarrow \mathcal{L}(\mathcal{P}(M))$ lifts to a flat connection on $\Pi(M)$ i.e., a morphism

$$\tau : \mathrm{Pair}(M) \circ\!\!\longrightarrow \Pi(M), \tag{41}$$

which is unique modulo germ equivalence. This is the flat direct connection on $\Pi(M)$ of Example 2.27 and Proposition 2.28.

The following assertion is a straightforward consequence of Proposition 2.30.

Corollary 2.32. *Let M be a smooth manifold endowed with an affine connection.*

(1) If ω is a principal connection on a principal bundle $P \to M$ and Γ is the direct connection on $\mathcal{G}(P) \rightrightarrows M$ induced by the parallel transport on P, then ω is flat as a principal connection if and only if Γ is flat as a direct connection.

(2) If ∇ is a linear connection on a vector bundle $E \to M$ and Γ is the direct connection on $\mathrm{Iso}(E) \rightrightarrows M$ induced by the parallel transport on E, then ∇ is flat as a linear connection if and only if Γ is flat as a direct connection.

3. Jet prolongation of bundles and groupoids

Jet prolongations of groupoids were first considered by Ehresman [Eh55] and later revisited by Kolár [Kol07]. In this section we briefly review jets of smooth functions and sections, which are coordinate free objects, and their coordinate dependent representation via Taylor polynomials. We recall the main notations and facts on jets of local sections of vector and principal bundles, referring to the book [KMS] by I. Kolář, P. Michor and J. Slovak for details. We then turn to jet prolongations of groupoids bisections. and discuss the functoriality of the jet prolongation of gauge groupoids (see eq. (73)) and their Lie algebroids (see eq. (88)). Specialising to frame groupoids of vector bundles, we further compare the jet prolongation of a frame groupoid of a vector bundle with the frame groupoid of the jet prolongation of a vector bundle.

Throughout this section we work in the category of smooth manifolds M with smooth local maps $f : M \to M'$. Following our previous conventions, we denote by $f : M \mathrel{*\!\!\longrightarrow} M'$ a smooth local map defined on some open subset of M, and call it simply a *local map*. Similarly, we denote by $f : M \mathrel{\overset{\sim}{*\!\!\longrightarrow}} M'$ a smooth local map between M and M' which is invertible with smooth inverse, that is, a diffeomorphism between two open subsets $U \subset M$ and $V \subset M'$, and call it simply a *locally defined diffeomorphism*.

3.1. **Jets and Taylor polynomials of smooth local functions.** Let M and M' be two smooth manifolds of dimension respectively d and d'. Given a local map $f : M \mathrel{*\!\!\longrightarrow} M'$ defined around a point $x \in M$, the n**-jet of** f **in** x, denoted by $j^n_x f$, is the equivalence class of local maps from M to M' having the same **contact of order** n of f in x, that is, the same value and the same derivatives in x up to order n. Jets of functions can be defined for any integer order $n \geqslant 0$, and $J^0(M, M') = M \times M'$.

Denote by $J^n_x(M, M')_y$ the space of n-jets of local maps $f : M \mathrel{*\!\!\longrightarrow} M'$ defined around x and such that $y = f(x)$. We further set

$$J^n_x(M, M') := \bigcup_{y \in M'} J^n_x(M, M')_y, \quad J^n(M, M') := \bigcup_{(x,y) \in M \times M'} J^n_x(M, M')_y.$$

As equivalence classes, jets are by definition independent of local coordinates in M and in M', but a representative of a jet involves derivatives which do depend on the choice of local coordinates.

Given a choice of coordinates on M and on M', the identification of a local map $f : M \mathrel{*\!\!\longrightarrow} M'$ fixing $f(x) = y$ to its local coordinates expression $\tilde{f} : \mathbb{R}^d \mathrel{*\!\!\longrightarrow} \mathbb{R}^{d'}$, fixing $\tilde{f}(0) = 0$, yields an isomorphism [KMS, §12.6] between $J^n_x(M, M')_y$ and the real vector space

$$L^n_{d,d'} := J^n_0(\mathbb{R}^d, \mathbb{R}^{d'})_0 \cong \bigoplus_{k=1}^{n} S^k((\mathbb{R}^d)^*) \otimes \mathbb{R}^{d'} \cong \overline{\mathbb{R}_n[X_1, ..., X_d]} \otimes \mathbb{R}^{d'} \tag{42}$$

of dimension $\left[\binom{d+n}{d} - 1\right] d'$ which prolongs the matrix space $L^1_{d,d'} \cong M_{d,d'}(\mathbb{R})$ for $n = 1$ and contains the matrix coefficents $a^j_\alpha = \frac{1}{|\alpha|!} \partial^\alpha \tilde{f}^j(0) \in \mathbb{R}$.

3.2. **Higher frame bundle and higher tangent bundle.** Jet composition is a key operation allowing us to express the invariance of jets under change of local coordinates. For two local maps $f : M \mathrel{*\!\!\longrightarrow} M'$ around x and $f' : M' \mathrel{*\!\!\longrightarrow} M''$ around $f(x) \in M'$, the **jet composition** of $j^n_x f$ and $j^n_{f(x)} f'$ is the n-jet [KMS, §12.3]

$$j^n_{f(x)} f' \circ j^n_x f = j^n_x (f' \circ f). \tag{43}$$

The jet composition is associative, unital, with a two-sided unit $1_x := j^n_x \mathrm{Id_M}$, and preserves the inversion, that is, if $h : M \overset{\sim}{\ast\!\!\rightarrow} M'$ is a locally defined diffeomorphism, then $(j^n_x h)^{-1} = j^n_{h(x)} h^{-1}$.

As a consequence, we can identify the **jets of locally defined diffeomorphisms** $h : M \overset{\sim}{\ast\!\!\rightarrow} M'$ with the invertible jets from M to M', that is, with the jet space

$$\mathrm{inv}J^n(M, M') := \text{set of invertible } n\text{-jets } j^n_x h \in J^n(M, M') \text{ with respect to jet composition,} \tag{44}$$

and consider the **n-jet group** or **n-differential group in dimension** d [KMS, §12.6]

$$\begin{aligned} GL^n_d(\mathbb{R}) &:= \mathrm{inv}L^n_{d,d} = \mathrm{inv}J^n_0(\mathbb{R}^d, \mathbb{R}^d)_0 \\ &\cong \text{group of } n\text{-jets at } 0 \text{ of locally defined diffeomorphisms } h : \mathbb{R}^d \overset{\sim}{\ast\!\!\rightarrow} \mathbb{R}^d \\ &\quad\ \text{preserving } 0, \text{ with jet composition.} \end{aligned} \tag{45}$$

We have $GL^0_d(\mathbb{R}) \cong \{1_{GL_d(\mathbb{R})}\}$ and the first jet group yields the general linear group $GL^1_d(\mathbb{R}) \cong GL_d(\mathbb{R})$, where a jet $j^1_0 h \in GL^1_d(\mathbb{R})$ is identified to the differential $dh_0 \in GL_d(\mathbb{R})$.

The effect on jet spaces of the choice of local coordinates on M is ruled by the **n-frame bundle of** M [KMS, §12.12] [Kol09, §1]

$$\begin{aligned} F^n M &:= \mathrm{inv}J^n_0(\mathbb{R}^d, M) \xrightarrow{\pi^n_0} M \\ &\cong \text{set of } n\text{-jets at } 0 \text{ of locally defined diffeomorphisms } \varphi : \mathbb{R}^d \overset{\sim}{\ast\!\!\rightarrow} M. \end{aligned} \tag{46}$$

This is a principal $GL^n_d(\mathbb{R})$-bundle with right action given by the jet composition $j^n_0\varphi \circ j^n_0 h = j^n_0(\varphi \circ h)$, where $\varphi : \mathbb{R}^d \overset{\sim}{\ast\!\!\rightarrow} M$ is a locally defined diffeomorphism around 0, which represents a choice of local coordinates on M around $\varphi(0) = x$, and $h : \mathbb{R}^d \overset{\sim}{\ast\!\!\rightarrow} \mathbb{R}^d$ is locally defined diffeomorphism preserving 0, which represents a change of local coordinates on M at x. We have $F^0 M \cong M \times \{1\}$ and the usual frame bundle corresponds to the 1-frame bundle $F^1 M \cong F(TM)$, since a jet $j^1_0\varphi \in F^1 M$ is equivalent to the pair $(\varphi(0), d\varphi_0) \in FM$ where $\varphi(0) = x \in M$ and where the differential $d\varphi_0 : \mathbb{R}^d \xrightarrow{\cong} T_x M$ determines a linear frame of TM at x.

The effect on jets of reading a local map $f : M \to M'$ in local coordinates on M, is governed on M' by the **n-tangent bundle of M' in dimension** d, also called the space of **n-velocities on M' in dimension** d [KMS, §12.8],

$$T^n_d M' := J^n_0(\mathbb{R}^d, M') \xrightarrow{\pi^n_0} M'. \tag{47}$$

The n-jet group $GL^n_d(\mathbb{R})$ acts on the left on $T^n_d M'$ by jet composition with the inverse diffeomorphism, that is $j^n_0 h \cdot j^n_0 \bar{f} = j^n_0(\bar{f} \circ h^{-1})$ where $\bar{f} : \mathbb{R}^d \ast\!\!\longrightarrow M'$ is a local map around 0 and $h : \mathbb{R}^d \overset{\sim}{\ast\!\!\rightarrow} \mathbb{R}^d$ is locally defined diffeomorphism preserving 0. In particular, in dimension $d = 1$, the **n-tangent bundle of** M

$$T^n M := J^n_0(\mathbb{R}, M) \xrightarrow{\pi^n_0} M, \tag{48}$$

is a jet prolongation of the usual tangent bundle $TM \cong J^1_0(\mathbb{R}, M)$ of vectors tangent to curves on M.

Combining the above ingredients yields a description of the jet bundle $J^n(M, M') \to M$ as the fibre bundle associated to the n-frame bundle of M with fibre given by n-tangent space of M' in dimension d, that is (see e.g. [KMS, §12.12])

$$\begin{aligned} J^n(M,M') &\cong F^nM \times_{GL^n_d(\mathbb{R})} T^n_d M' \\ &= \{[j^n_0\varphi, j^n_0\bar{f}] \mid \varphi : \mathbb{R}^d \overset{\sim}{\twoheadrightarrow} M,\ \bar{f} : \mathbb{R}^d \twoheadrightarrow M'\} \\ j^n_x f &\mapsto [j^n_0\varphi, j^n_0\bar{f}] \quad \text{such that } \bar{f} = f\circ\varphi. \end{aligned} \tag{49}$$

The bundle projection $J^n(M,M') \to M'$ is then inherited from that of the fibre $T^n_d M'$.

3.3. **Jet prolongation of fibre bundles.** The n**-jet bundle** of a fibre bundle $\pi : E \to M$ is the subset

$$J^nE := \text{set of } n\text{-jets of smooth local sections } f : M \twoheadrightarrow E \text{ of } E \tag{50}$$

of the jet space $J^n(M,E)$, which enjoys the following properties:

- J^nE is a closed submanifold of $J^n(M,E)$ [KMS, §12.16],
- the anchor restricts to a bundle map which coincides with the target projection, namely $\pi^n_0 : J^nE \to M \times_M E = E$,
- the source $\pi^n : J^nE \to M$ commutes with π^n_0 and π, and is locally trivial [Sa09, §6.2],
- the partial jet projections descend to the spaces J^kE with $1 \leqslant k \leqslant n-1$.

The (source) jet projection $\pi^n : J^nE \to M$ is a fibre bundle, while the target jet bundle $\pi^n_0 : J^nE \to E$ is again a filtered tower of affine bundles

$$J^nE \overset{\pi^n_{n-1}}{\to} J^{n-1}E \overset{\pi^{n-1}_{n-2}}{\to} \cdots \overset{\pi^2_1}{\to} J^1E \overset{\pi^1_0}{\to} J^0E = E, \tag{51}$$

A smooth map $\phi : E \longrightarrow E'$ between two fibre bundles $\pi : E \to M$, $\pi' : E' \to M'$ induces a smooth map between their n-jet prolongation

$$J^n\phi : (\pi^n : J^nE \to M) \longrightarrow \left(\pi'^n : J^nE' \to M'\right),\ j^n_x\sigma \mapsto J^n\phi(j^n_x\sigma) := j^n_x(\phi\circ\sigma). \tag{52}$$

If $\pi : E \to M$ is a vector bundle of rank r with typical fibre given by a vector space $V \cong \mathbb{R}^r$, then the source jet projection $\pi^n : J^nE \to M$ is also a vector bundle [KMS, §12.17]. The fibre of J^nE is modelled (in the affine filtered sense and for given local coordinates) on

$$J^n_0(\mathbb{R}^d, V) = T^n_d V, \tag{53}$$

where d is the dimension of M. Equivalently, the fibre is modelled on the real vector space

$$\begin{aligned} P^n_{d,r} := T^n_d\,\mathbb{R}^r = J^n_0(\mathbb{R}^d,\mathbb{R}^r) &\cong \mathbb{R}^r \oplus L^n_{d,r} \\ &\cong \bigoplus_{k=0}^n S^k((\mathbb{R}^d)^*)\otimes\mathbb{R}^r \cong \mathbb{R}_n[X_1,...,X_d]\otimes\mathbb{R}^r \end{aligned} \tag{54}$$

of dimension $\binom{d+n}{d}r$, that is, the space of $\mathbb{R}^r$-valued polynomials in d variables.

3.4. **Jet prolongation of structure groups.** We saw that the jet bundle of a vector bundle $E \to M$ with fibre $\mathbb{R}^r$ bundle $J^nE \to M$ has fibre $P^n_{d,r} = T^n_d(\mathbb{R}^r)$ (see eq. (54)). Its structure group is given by the subgroup of $GL(P^n_{d,r})$ which hosts the transition functions $\tilde{g}_{\alpha\beta}$ of J^nE, defined on the intersection $U_{\alpha\beta} = U_\alpha \cap U_\beta$ of overlapping charts in a trivializing atlas $\{U_\alpha \subset M\}$ of E.

Such maps are naturally described as jets of the transitions functions $g_{\alpha\beta}$ of E. If E has structure group $G \subset GL_r(\mathbb{R})$, then the structure group of J^nE contains the jets in 0 of local maps $g : \mathbb{R}^d \twoheadrightarrow G$, that is, the jets $j^n_0 g$ in the n-tangent space $T^n_d G$ (defined as in eq. (47) with $M' = G$)), together with the jets in 0 of locally defined diffeomorphisms $h : \mathbb{R}^d \overset{\sim}{\twoheadrightarrow} \mathbb{R}^d$

fixing the point $0 \in \mathbb{R}^d$, that is, the jets $j_0^n h$ in the n-jet group $GL_d^n(\mathbb{R})$, cf. eq. (45). Since G is a Lie group, the space $T_d^n G$ is also a Lie group, with the operation

$$j_0^n g \cdot j_0^n g' = j_0^n (gg') \tag{55}$$

induced on two smooth local maps $g, g' : \mathbb{R}^d \ast\!\!\rightarrow G$ by the pointwise product gg' in G. Furthermore, the action of the diffeomorphism h on the variable of g becomes a right action of $GL_d^n(\mathbb{R})$ on $T_d^n G$ by jet composition. Both contributions, from $j_0^n h$ in $GL_d^n(\mathbb{R})$ and from $j_0^n g$ in $T_d^n G$, must be taken into account to describe the jets of transition functions of E.

Finally, the structure group of $J^n E$ reduces to the the semidirect product [KMS, §15.2]

$$W_d^n G := GL_d^n \ltimes T_d^n G = \left\{ \, (j_0^n h, j_0^n g) \mid h \,\tilde{\circlearrowleft}\, \mathbb{R}^d \overset{g}{\ast\!\!\longrightarrow} G \, \right\}, \tag{56}$$

called the **n-jet prolongation of G in dimension** d, with usual semidirect group law

$$\begin{aligned} \big(j_0^n h, j_0^n g\big) \cdot \big(j_0^n h', j_0^n g'\big) &:= \Big(j_0^n h \circ j_0^n h', \big(j_0^n g \circ j_0^n h'\big) j_0^n g'\Big) \\ &= \Big(j_0^n (h \circ h'), j_0^n \big((g \circ h') g'\big)\Big), \end{aligned} \tag{57}$$

which makes use of both the jet composition (43) and the group operation (55).

The transition functions of E can also be seen as G-equivariant maps

$$U_{\beta|_{U_{\alpha\beta}}} \times G \to U_{\alpha|_{U_{\alpha\beta}}} \times G, \ (x, 1) \mapsto \big(x, g_{\alpha\beta}(x)\big),$$

and the jet prolongation group $W_d^n G$ can be viewed as the jet space [KMS, §15.2]:

$$\begin{aligned} W_d^n G \cong{} & \text{set of } n\text{-jets at } (0,1) \in \mathbb{R}^d \times G \text{ of } G\text{-equivariant locally defined} \\ & \text{diffeomorphisms } \phi : \mathbb{R}^d \times G \overset{\sim}{\ast\!\!\rightarrow} \mathbb{R}^d \times G \text{ which preserve } 0 \in \mathbb{R}^d. \end{aligned} \tag{58}$$

Here, a pair $(j_0^n h, j_0^n g) \in GL_d^n \ltimes T_d^n G$ is mapped to the jet $j_{(0,1)}^n \phi$ of the G-equivariant locally defined diffeomorphism defined by $\phi(x,1) = (h(x), g(x))$ and therefore, $\phi(x,a) = (h(x), g(x)a)$ for any $a \in G$.

Example 3.1. In particular, if $G = GL_r(\mathbb{R})$, there is an inclusion of groups

$$W_d^n GL_r(\mathbb{R}) = GL_d^n(\mathbb{R}) \ltimes T_d^n GL_r(\mathbb{R}) \hookrightarrow GL(P_{d,r}^n), \tag{59}$$

which assigns to the pair of jets $(j_0^n h, j_0^n g)$ the linear invertible map on $P_{d,r}^n = J_0^n(\mathbb{R}^d, \mathbb{R}^r)$ acting on the jet $j_0^n \bar{f} = \bar{f}(0) + j_0^n \tilde{f}$ of a smooth local function $\tilde{f} : \mathbb{R}^d \ast\!\!\rightarrow \mathbb{R}^r$ (the local coordinates expression of a section of E) in adding the term $g(0) \cdot \bar{f}(0)$ to the n-jet at 0 of the function

$$(h, g) \cdot \tilde{f} := (g \cdot \tilde{f}) \circ h^{-1} = (g \circ h^{-1}) \cdot (\tilde{f} \circ h^{-1})$$

explicitly given by $\big((h,g) \cdot \tilde{f}\big)(x) = g\big(h^{-1}(x)\big) \cdot \tilde{f}\big(h^{-1}(x)\big)$ on $x \in \mathbb{R}^d$.

□

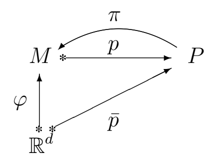
3.5. **Jet prolongation of principal bundles.** Let $\pi : P \to M$ be a principal bundle with fibre given by a Lie group G. According to eqs. (49) and (53), the jet bundle $J^n P \to M$ of smooth local sections of P has typical fibre modelled on the higher tangent Lie group $T_d^n G$. However, $J^n P$ is *not* a principal $T_d^n G$-bundle since we need a jet prolongation of P

with structure group given by the jet prolongation group $W^n_d G = GL^n_d \ltimes T^n_d G$. Instead, we consider the jet space [KMS, §15.4], [Kol09, Formula (33)]

$$W^n P := F^n M \overset{\pi}{\times} J^n P = \{(j^n_0\varphi, j^n_x p) \mid \varphi(0) = x\}, \tag{60}$$

which describes the jets of local sections of P together with a choice of local coordinates allowing to realise its transition functions as actual elements of a matrix group. An element of $W^n P$ can equivalently be represented as

$$(j^n_0\varphi, j^n_x p) = j^n_0 \bar{p} \tag{61}$$



where $\bar{p} = p \circ \varphi : \mathbb{R}^d \ast\!\!\to P$ is such that $\bar{p}(0) = p(x)$.

The Lie group $T^n_d G$ acts on this jet space by

$$\big(j^n_0\varphi, j^n_x p\big) \cdot j^n_0 g := \Big(j^n_0\varphi, j^n_x\big(p(g \circ \varphi^{-1})\big)\Big),$$

where $g : \mathbb{R}^d \ast\!\!\to G$ is a smooth local map defined around 0. It can be prolonged to an action of the jet group $W^n_d G = GL^n_d \ltimes T^n_d G$ as

$$\big(j^n_0\varphi, j^n_x p\big) \cdot \big(j^n_0 h, j^n_0 g\big) := \Big(j^n_0(\varphi \circ h), j^n_x\big(p(g \circ (\varphi \circ h)^{-1})\big)\Big),$$

which is proved to be a principal action. The bundle $W^n P$ is therefore a principal $W^n_d G$-bundle, called the **n-principal prolongation of P**.

In analogy with the alternative presentation of the structure group $W^n_d G$ given in (58), the bundle $W^n P$ can alternatively be defined as the jet space [KMS, §15.3]

$$W^n P = \begin{array}{l}\text{set of } n\text{-jets at } (0,1) \text{ of bundle automorphisms } \mathbb{R}^d \times G \ast\!\!\to P \\ \text{above base maps } \mathbb{R}^d \ast\!\!\to M\end{array}, \tag{62}$$

with bundle projection to M given by the projection to M of the jet target to P.

One can further check that a morphism $\phi : P_1 \to P_2$ of two principal bundles induces a morphism of principal bundles $W^n\phi : W^n P_1 \to W^n P_2$.

3.6. **Associated jet bundles and reduction of jet groups.** Let $P \to M$ be a principal G-bundle and let $E = P \times_G V$ be an associated fibre bundle with fibre V. It is shown in [KMS, §15.5] that $J^n E$ is the fibre bundle associated to $W^n P$ with fibre $T^n_d V$, that is,

$$J^n E \cong W^n P \times_{W^n_d G} T^n_d V. \tag{63}$$

Since $E = P \times_G V$, we have

$$T^n_d E \cong T^n_d P \times_{W^n_d G} T^n_d V,$$

and hence

$$J^n E \cong F^n M \overset{\pi}{\times}_{GL^n_d(\mathbb{R})} T^n_d E \cong F^n M \overset{\pi}{\times}_{GL^n_d(\mathbb{R})} T^n_d P \times_{W^n_d G} T^n_d V \cong W^n P \times_{W^n_d G} T^n_d V.$$

Example 3.2. Let us consider a vector bundle E of rank r, with typical fibre $\mathbb{R}^r$ and structure group $GL_r(\mathbb{R})$, from which we can build three interesting jet bundles:

(1) On one hand, the jet bundle $J^n E$ is a vector bundle with typical fibre $T^n_d\,\mathbb{R}^r = P^n_{d,r}$, and its frame bundle $F(J^n E)$ can be viewed as an associated principal $GL(P^n_{d,r})$-bundle:

$$J^n E \cong F(J^n E) \times_{GL(P^n_{d,r})} P^n_{d,r}.$$

(2) On the other hand, the principal $GL_r(\mathbb{R})$-bundle associated to E is the frame bundle FE such that $E \cong FE \times_{GL_r(\mathbb{R})} \mathbb{R}^r$. The existence of its principal jet prolongation W^nFE verifying the identity (63), namely $J^nE \cong W^nFE \times_{W^n_d GL_r(\mathbb{R})} P^n_{d,r}$, says that the structure group of J^nE can always be reduced to $W^n_d GL_r(\mathbb{R}) \subsetneq GL(P^n_{d,r})$, and therefore we have

$$F(J^nE) \cong W^nFE \times_{W^n_d GL_r(\mathbb{R})} GL(P^n_{d,r}). \tag{64}$$

(3) Finally, if the structure group of E can be reduced to $G \subset GL_r(\mathbb{R})$, so that there is a principal G-bundle $P \to M$ such that $E \cong P \times_G \mathbb{R}^r$, then by eq. (63) the structure group of J^nE can be further reduced to $W^n_d G \subset W^n_d GL_r(\mathbb{R})$ and we have

$$J^nE \cong W^nP \times_{W^n_d G} P^n_{d,r},$$

and hence

$$W^nFE \cong W^nP \times_{W^n_d G} W^n_d GL_r(\mathbb{R}), \tag{65}$$
$$F(J^nE) \cong W^nP \times_{W^n_d G} GL(P^n_{d,r}). \tag{66}$$

Note that there is a proper inclusion of groups

$$W^n_d G \subsetneq W^n_d GL_r(\mathbb{R}) \subsetneq GL(P^n_{d,r}),$$

and therefore a proper inclusion of bundles

$$W^nP \subsetneq W^nFE \subsetneq F(J^nE). \tag{67}$$

3.7. **Jet prolongation of groupoids.** Following [Eh55], [Kol07, §1], [Me17, §4.5], we recall the definition of the jet prolongation of a Lie groupoid $\mathcal{G} \rightrightarrows M$ by means of local bisections defined in §1.9. The **n-jet prolongation of $\mathcal{G}$** is the jet space

$$J^n\mathcal{G} := \text{set of } n\text{-jets of local bisections } \sigma : M \ast\!\!\to \mathcal{G} \quad , \tag{68}$$

together with the structure of a Lie groupoid on M induced by that of $\mathcal{G}$:

(1) source and target maps $s^n, t^n : J^n\mathcal{G} \to M$ given respectively by the surjective submersions $s^n(j^n_x\sigma) = x$ and $t^n(j^n_x\sigma) = t\big(\sigma(x)\big)$ for any $x \in M$,
(2) the multiplication $j^n_{x'}\sigma' j^n_x\sigma = j^n_x\big(\sigma' \bullet \sigma\big)$ defined if only if $x' = \varphi_\sigma(x)$, where $\bullet$ is the semidirect product of bisections given in Section 1.9,
(3) unit $u^n(x) \equiv 1_x = j^n_x u$, where $u : M \to \mathcal{G}$ is the unit map of $\mathcal{G}$,
(4) inverse $(j^n_x\sigma)^{-1} = j^n_{\varphi_\sigma^{-1}(x)}\sigma^{-1}$ where σ^{-1} is the inverse local bisection as given above.

Since $\mathcal{G}$ is a fibered manifold on M by the source map, the jet prolongation $J^n\mathcal{G}$ enjoys the same properties as the jet bundle of a fibre bundle described in Section 3.3. In particular:

- $J^n\mathcal{G}$ is a closed submanifold of the jet space $J^n(M, \mathcal{G})$ [KMS, §12.16],
- the natural projection $\pi^n_0 : J^n\mathcal{G} \to \mathcal{G},\ j^n_x\sigma \mapsto \sigma(x)$ is locally trivial,
- the jet projections $\pi^k_{k-1} : J^k\mathcal{G} \to J^{k-1}\mathcal{G},\ j^k_x\sigma \mapsto j^{k-1}_x\sigma$, for any $1 \leqslant k \leqslant n$, give a filtered tower of affine bundles [KMS, §12.11] similar to (51)

$$J^n\mathcal{G} \to J^{n-1}\mathcal{G} \to \cdots \to J^1\mathcal{G} \to J^0\mathcal{G} = \mathcal{G}, \tag{69}$$

carrying each a Lie groupoid structure over M.

Furthermore, the natural projection $\pi_0^n : J^n\mathcal{G} \to \mathcal{G}$ is a groupoid morphism. In fact, for any σ, σ' in $\mathcal{B}_{\text{loc}}(\mathcal{G})$ and for any x in M such that σ' is defined in $x' = \varphi_\sigma(x)$, we have

$$\begin{aligned}\pi_0^n(j_{x'}^n\sigma'\, j_x^n\sigma)) &= \pi_0^n(j_x^n(\sigma' \bullet \sigma)) = (\sigma' \bullet \sigma)(x) \\ &= \sigma'(x')\,\sigma(x) = \pi_0^n(j_{x'}^n\sigma')\,\pi_0^n(j_x^n\sigma).\end{aligned}$$

Given a smooth map $\phi : \mathcal{G} \longrightarrow \mathcal{G}'$ between two Lie groupoids over M, and using the map $\mathcal{B}_{\text{loc}}(\phi)$ of eq. (19), we define a smooth map between their n-jet prolongation Lie groupoids by setting

$$J^n\phi : J^n\mathcal{G} \longrightarrow J^n\mathcal{G}',\ j_x^n\sigma \mapsto J^n\phi(j_x^n\sigma) := j_x^n(\phi \circ \sigma). \tag{70}$$

If ϕ is a groupoid morphism, then $J^n\phi$ is also a groupoid morphism, because $\mathcal{B}_{\text{loc}}(\phi)$ is a morphism of pseudo-groups (cf. Paragraph 1.9).

Thanks to the locality of the notion of jet, eq. (70) actually extends to any local map $\phi : \mathcal{G} \ast\!\!\longrightarrow \mathcal{G}'$ defined on an open neighborhood $\mathcal{U}$ of the diagonal $u(M)$ in $\mathcal{G}$, where u is the unit on $\mathcal{G}$. Explicitly, by (19), the map ϕ induces a map

$$\mathcal{B}_U(\mathcal{G}) \longrightarrow \mathcal{B}_{\varphi_\sigma^{-1}(U)}(\mathcal{G}'),\ \sigma \mapsto \phi \circ \sigma, \tag{71}$$

for any open subset $U \subset M$ chosen small enough so that $u(U)$ lies in $\mathcal{U}$ and we build a local map $J^n\phi : J^n\mathcal{G} \ast\!\!\longrightarrow J^n\mathcal{G}'$ defined on any open neighborhood $\mathcal{U}^n$ of the diagonal $u^n(M) = \{j_x^n u,\ x \in M\}$ in $J^n\mathcal{G}$ such that $\pi_0^n\big(\mathcal{U}^n\big) \subset \mathcal{U}$ as follows.

An element in $\mathcal{U}^n$ is the jet $j_x^n\sigma$ of a local bisection σ of $\mathcal{G}$ defined in a neighborhood of x such that $\pi_0^n(j_x^n\sigma) = \sigma(x) \in \mathcal{U}$. The domain U_x of σ should be chosen small enough so that the image $\sigma(U_x) \subset \mathcal{G}$ is contained in $\mathcal{U}$. Note that with the Fréchet topology induced by the supremum norm in all derivatives of order no larger than n on compact subsets, the union $\bigcup_{x\in M}\mathcal{U}_{j_x^n\sigma}$ of open neighborhoods of the point $u^n(x) = j_x^n u$ in $J^n\mathcal{G}$ gives rise to a neighborhood $\mathcal{U}^n$ of the diagonal in $J^n\mathcal{G}$ with the property that $\pi_0^n\big(\mathcal{U}^n\big)$ lies in $\mathcal{U}$.

We set

$$J^n\phi(j_x^n\sigma) := j_x^n(\phi \circ \sigma\big|_{U_x'}). \tag{72}$$

Proposition 3.3. [Kol07] *The n-jet prolongation of a gauge groupoid $\mathcal{G}(P) \rightrightarrows M$ is isomorphic to the gauge groupoid of the n-jet principal bundle W^nP, namely*

$$J^n\mathcal{G}(P) \cong \mathcal{G}(W^nP). \tag{73}$$

Furthermore, the jet prolongation of a morphism $\phi : \mathcal{G}(P_1) \to \mathcal{G}(P_2)$ yields a morphism $j^n\phi : \mathcal{G}(W^nP_1)$ to $\mathcal{G}(W^nP_2)$ of gauge groupoids.

Proof. This result is proved in [Kol07, §4, after eq. (15)] for all natural functors. □

Examples 3.4. Proposition 3.3 gives rise to several examples:

(1) For $P = M \times G$ we have $\mathcal{G}(M \times G) = M \times G \times M$, and

$$J^n(M \times G \times M) \simeq \mathcal{G}(W^n(M \times G)) \simeq \big(F^nM \times_{GL_d^n} F^nM\big) \times T_d^nG, \tag{74}$$

(2) Specialising to the gauge groupoid of the trivial bundle $P = M \times \{1\}$ yields the pair groupoid $\text{Pair}(M) = \mathcal{G}(M \times \{1\})$. Since $T_d^n(\{1\}) = \{1\}$, eq. (74) with $G = \{1\}$ yields

$$J^n\,\text{Pair}(M) = \text{inv}J^n(M, M) \cong \mathcal{G}(F^nM) = F^nM \times_{GL_d^n(\mathbb{R})} F^nM, \tag{75}$$

which is confirmed by $W^n(M \times \{1\}) = F^nM$ (cf. eq. (46)).

For $n = 1$, since $GL^1_d = GL_d(\mathbb{R})$ and $F^1M = F(TM)$ is a frame bundle, the 1-jet groupoid is a frame groupoid [Me17, Example 1.12]

$$J^1\,\mathrm{Pair}(M) \cong \mathcal{G}(FM) \cong \mathrm{Iso}(TM). \tag{76}$$

(3) The frame groupoid $\mathrm{Iso}(E)$ of a vector bundle E with rank r is the gauge groupoid of the frame bundle FE with fibre $GL_r := GL_r(\mathbb{R})$. It follows from eq.(73) that

$$J^n\mathrm{Iso}(E) \cong \mathcal{G}(W^nFE) = W^nFE \times_{W^n_d GL_r} W^nFE. \tag{77}$$

(4) Suppose that the vector bundle E of rank r admits a reduction of its structure group to a subgroup $G \subset GL_r(\mathbb{R})$ and further to the trivial group $\{1\} \subset GL_r(\mathbb{R})$ (which forces E to be trivializable, i.e. $E \cong M \times \mathbb{R}^r$). Then its frame bundle FE admits a reduction first to a principal G-bundle P, and further to the trivial principal bundle $M \times \{1\}$, which yields a sequence of gauge subgroupoids

$$\mathcal{G}(M \times \{1\}) = \mathrm{Pair}(M) \subset \mathcal{G}(P) \subset \mathcal{G}(FE) = \mathrm{Iso}(E).$$

The jet bundle J^nE has structure group $GL(P^n_{d,r})$, frame bundle $F(J^nE)$ and frame groupoid $\mathrm{Iso}(J^nE)$. Applying the n-jet prolongation to all ingredients yields a sequence of subgroups

$$\begin{aligned} W^n_d\{1\} = GL^n_d(\mathbb{R}) \subset W^n_d G &= GL^n_d(\mathbb{R}) \ltimes T^n_d G \\ \subset W^n_d GL_r &= GL^n_d(\mathbb{R}) \ltimes T^n_d GL_r \\ &\subset GL(P^n_{d,r}), \end{aligned} \tag{78}$$

which in turn induces a sequence of reduced principal bundles

$$\begin{aligned} W^n(M \times \{1\}) = F^nM \subset W^nP &= F^nM \overset{\pi}{\times} J^nP \\ \subset W^nFE &= F^nM \overset{\pi}{\times} J^nFE \\ &\subset F(J^nE), \end{aligned} \tag{79}$$

where P is the reduced frame bundle FE with structure group G. We finally get a sequence of subgroupoids

$$\begin{aligned} \mathcal{G}(F^nM) = J^n\,\mathrm{Pair}(M) \subset \mathcal{G}(W^nP) &= J^n\mathcal{G}(P) \\ \subset \mathcal{G}(W^nFE) &= J^n\mathrm{Iso}(E) \\ \subset \mathcal{G}(F(J^nE)) &= \mathrm{Iso}(J^nE). \end{aligned} \tag{80}$$

□

3.8. **The frame groupoid of a jet prolonged vector bundle.** The frame groupoid $\mathrm{Iso}(J^nE)$ of the jet bundle J^nE of a vector bundle $\pi : E \to M$ plays a central role in the context of regularity structures. It is the gauge groupoid of the frame bundle $F(J^nE)$, which is a principal $GL(P^n_{d,r})$-bundle and $\mathrm{Iso}(J^nE)$ contains $J^n\mathrm{Iso}(E) = \mathcal{G}(W^nFE)$ as a proper subgroupoid.

Indeed,

$$W^n_d GL_r = GL^n_d(\mathbb{R}) \ltimes T^n_d GL_r \subset GL(P^n_{d,r}), \quad W^nFE = F^nM \overset{\pi}{\times} J^nFE \subset F(J^nE)$$

and by eqs. (64) and (73) we have

$$\begin{aligned} \mathrm{Iso}(J^nE) &\cong \mathcal{G}(F(J^nE)) = F(J^nE) \times_{GL(P^n_{d,r})} F(J^nE) \\ &\cong W^nFE \times_{W^n_dGL_r} F(J^nE) \\ &\cong W^nFE \times_{W^n_dGL_r} GL(P^n_{d,r}) \times_{W^n_dGL_r} W^nFE. \end{aligned} \tag{81}$$

This shows that $J^n\mathrm{Iso}(E)$ is a proper subgroupoid of $\mathrm{Iso}(J^nE)$, with inclusion given by

$$[A,B]_\sim \mapsto [A,1,B]_\sim \qquad \text{for } A,B \in W^nFE,$$

where 1 is the unit in the group $GL(P^n_{d,r})$ and where the equivalence relations are

$$(AH,B) \sim (A,B\,H^{-1}) \qquad \text{and} \qquad (AH,G,BH') \sim (A,HG(H')^{-1},B)$$

for any $A,B \in W^nFE$, $G \in GL(P^n_{d,r})$ and $H,H' \in W^n_dGL_r$.

3.9. **Jet prolongation of Lie algebroids.** The n-jet bundle of a Lie algebroid $\mathcal{L} \to TM$ is the vector bundle $J^n\mathcal{L} \to M$ of its underlying vector bundle $\mathcal{L} \to M$. It turns out [Ku15, Proposition 1] [Me17, Example 6.11] that this jet bundle is again a Lie algebroid $J^n\mathcal{L} \to TM$, with anchor and bracket given by

$$a(j^n_xX) = a(X_x) \qquad \text{and} \qquad [j^n_xX, j^n_xY] = j^n_x([X,Y]_{\mathcal{L}}) \tag{82}$$

for any smooth sections X,Y of $\mathcal{L}$ and any x in M.

If $\phi : \mathcal{L} \to \mathcal{L}'$ is a smooth map between Lie algebroids, there exists a jet prolongation

$$J^n\phi : J^n\mathcal{L} \to J^n\mathcal{L}',\ j^n_xX \mapsto J^n\phi(j^n_xX) := j^n_x(\phi \circ X). \tag{83}$$

If ϕ is a morphism of Lie algebroids, by (82) the map $J^n\phi$ is also a morphism of Lie algebroids. Hence J^n is a functor on the category of Lie algebroids.

The n-jet prolongation of the Atiyah algebroid $A(P) \to M$ of a principal bundle $P \to M$ is isomorphic to the Atiyah algebroid of its n-jet principal bundle W^nP [Kol08, §2 Eq. (8)]

$$J^nA(P) \cong A(W^nP). \tag{84}$$

Specialising (83) to $\mathcal{L} = A(P)$, we see that the n-th jet prolongation defined in eq. (83) applied to the morphism of vector bundles $A(\phi) : A(P_1) \to A(P_2)$, with $P_1 \to M$ and $P_2 \to M$ two principal bundles, gives rise to the map $J^nA(\phi) : J^nA(P_1) \to A(P_2)$, which coincides with $A(W^n\phi) : A(W^nP_1) \to A(W^nP_2)$.

Example 3.5. In particular:

(1) If $P = M \times \{1\}$ is the trivial principal bundle with trivial fibre, we have $A(M \times \{1\}) = TM$ and $W^n(M \times \{1\}) = F^nM$, therefore [Kol09, Proposition 1], [Kol08, §2 Eq. (6)]

$$J^nTM \cong A(F^nM). \tag{85}$$

The case $n = 1$ is particularly important for connections, and gives a sequence of key isomorphisms induced by eqs. (85), (9), (76) and Example 1.11 (4):

$$J^1TM \cong A(FM) \cong \mathcal{L}(J^1\,\mathrm{Pair}(M)) \cong \mathcal{L}(\mathrm{Iso}(TM)) \cong \mathrm{Der}(TM). \tag{86}$$

(2) If $P = M \times G$ is the trivial principal G-bundle, then $A(M \times G) = TM \oplus (M \times \mathfrak{g})$ and $W^n(M \times G) = F^nM \times^\pi T^n_dG$, therefore

$$J^n\big(TM \oplus (M \times \mathfrak{g})\big) \cong A(F^nM \times^\pi T^n_dG). \tag{87}$$

(3) If $P = FE$ is the frame bundle of a vector bundle $E \to M$, then $A(P) = \mathrm{Der}(E)$ is the Lie algebroid of derivations of E and $W^n(FE) \cong F^nM \times^\pi J^nFE$, therefore

$$J^n\mathrm{Der}(E) \cong A\big(F^nM \times^\pi J^nFE\big).$$

□

If $\mathcal{G}$ is a Lie groupoid and $\mathcal{LG}$ its Lie algebroid, there is an isomorphism of Lie algebroids [Ku15, Proposition 1 and Theorem 1], [Me17, Example 9.5 (e)]

$$\mathcal{L}\big(J^n\mathcal{G}\big) \cong J^n\mathcal{L}(\mathcal{G}). \tag{88}$$

Furthermore, the n-th jet prolongation $j^n\phi$ of a morphism $\phi : \mathcal{G}_1 \to \mathcal{G}_2$ of two groupoids induces by differentiation a morphism on their n-th jet prolongations $D(j^n\phi)|_M : \mathcal{L}(J^n\mathcal{G}_1) \to \mathcal{L}(J^n\mathcal{G}_2)$.

Example 3.6. (1) For $P = M \times \{1\}$, we have on one side $\mathcal{G}(M \times \{1\}) = \mathrm{Pair}(M)$ hence $J^n\mathcal{G}(M \times \{1\}) = J^n\,\mathrm{Pair}(M) = \mathcal{G}(F^nM)$ by eq. (75), and on the other side $\mathcal{L}(\mathrm{Pair}(M)) = TM$ hence $J^n\mathcal{L}(\mathrm{Pair}(M)) = J^nTM$, giving a sequence of isomorphisms [Kol08, §2. (6)]

$$\mathcal{L}(J^n\,\mathrm{Pair}(M)) \cong \mathcal{L}(\mathcal{G}(F^nM)) \cong A(F^nM) \cong J^nTM. \tag{89}$$

The case $n = 1$ reproduces the isomorphisms of eq. (86).

(2) For $P = M \times G$ we have $\mathcal{G}(M \times G) = M \times G \times M$, $\mathcal{L}(M \times G \times M) = TM \oplus (M \times \mathfrak{g})$ and (by eq. (74)) $J^n(M \times G \times M) \simeq \big(F^nM \times_{GL^n_d} F^nM\big) \times T^n_dG$ so that

$$\mathcal{L}\big(J^n(M \times G \times M)\big) \cong J^n\big(TM \oplus (M \times \mathfrak{g})\big),$$

which is consistent with eq. (87).

(3) If $P = FE$ if a frame bundle, then $\mathcal{G}(FE) = \mathrm{Iso}(E)$ is the frame groupoid of E, $\mathcal{L}(\mathrm{Iso}(E)) = \mathrm{Der}(E)$ and we have

$$\mathcal{L}\big(J^n\mathrm{Iso}(E)\big) \cong J^n\mathrm{Der}(E).$$

□

4. Direct connections on jet groupoids

In this section, we build direct connections on jet prolongations $J^n\mathcal{G}$ of Lie groupoids from a direct connection Γ on a Lie groupoid $\mathcal{G}$, by means of an affine connection on the underlying manifold. Building blocks of the construction are the exponential (local) bisection (Definition 4.3) and the related exponential direct connection Δ^M in eq. (97) on the frame groupoid $\mathcal{G}(FM) \simeq \mathrm{Iso}(TM)$, both of which use the parallel transport on TM induced by the affine connection on M. Taking jets of the exponential bisection gives rise to the exponential direct connection $\Delta_M^{(n)}$ (eq. (96) in Definition 4.4) on the jet prolongation $J^n\,\mathrm{Pair}(M)$ of the pair groupoid $\mathrm{Pair}(M)$ of M. In Theorem 4.5 we prove that the exponential direct connection $\Delta_M^{(n)}$ is a jet prolongation of Δ^M and in Theorem 4.6 that the infinitesimal connection of $\Delta_M^{(n)}$ on $J^n\,\mathrm{Pair}(M)$ is the exponential n-th order prolongation $\delta_M^{(n)} : TM \longrightarrow \mathcal{L}(J^n\,\mathrm{Pair}(M)) \cong J^nTM$ (eq. (99)) of the affine connection on M used in [Kol09, §5] to build infinitesimal connections on jet prolongations of groupoids. A similar construction yields a direct connection $\Gamma^{(n)}$ on the jet prolongation $J^n\mathcal{G}$ of a general Lie groupoid $\mathcal{G}$ from a direct connection Γ on $\mathcal{G}$, see eq. (103) in Definition 4.7, which gives back eq. (96) when $\mathcal{G} = \mathrm{Pair}(M)$. Corollary 4.8 shows that $\Gamma^{(n)}$, which yields an n-th order prolongation of Γ, factorises through $\Delta_M^{(n)}$. In Theorem 4.11, we show that any flat connection on the jet prolongation $J^n\mathcal{G}$ of a Lie groupoid over a flat manifold, factorises through $\Delta_M^{(n)}$. Direct connections on the frame groupoid $\mathrm{Iso}(J^nE)$ of a jet bundle are of special interest in the context of regularity structures. This is the gauge groupoid of the frame bundle $F(J^nE)$ with structure group $GL(P^n_{d,r})$, but it is *not* the jet prolongation of a groupoid. While this section mainly focusses on direct connections on *jet groupoids of frame bundles*, we dedicate §4.7 to *frame groupoids of jet bundles*. Inspired by [DDD19], we build a direct connection $\tilde{\Gamma}^{(n)}$ on $\mathrm{Iso}(J^nE)$, which is *not a jet prolongation* of Γ by means of local Taylor expansions, and compare it with $\Gamma^{(n)}$ in Proposition 4.18.

4.1. Higher order direct connections.

Let $\mathcal{G} \rightrightarrows M$ be a Lie groupoid with n-jet prolongation groupoid $J^n\mathcal{G}$.

Definition 4.1. We call ***n*-th order direct connection on** $\mathcal{G}$ a direct connection on the n-jet groupoid $J^n\mathcal{G}$, that is, a local map

$$\Sigma : \mathrm{Pair}(M) \ast\!\!\longrightarrow J^n\mathcal{G}, \tag{90}$$

such that

(1) $\Sigma(x,y) \in (J^n\mathcal{G})^x_y$ for all $(x,y) \in \mathcal{U}_\Delta$,
(2) for all $x \in M$, $\Sigma(x,x) = u^n(x)$ the unit in $(J^n\mathcal{G})^x_x$.

If $\pi^n_0 : J^n\mathcal{G} \to J^0\mathcal{G} = \mathcal{G}$ is the jet projection described in §3.7, the composite map

$$\Sigma_0 = \pi^n_0 \circ \Sigma : \mathrm{Pair}(M) \ast\!\!\longrightarrow \mathcal{G} \tag{91}$$

is a direct connection on $\mathcal{G}$, that we shall call **0-th order projection of** Σ.

Viceversa, given a direct connection Γ on $\mathcal{G}$, we call ***n*-th order prolongation of** Γ any direct connection $\Gamma^{(n)}$ on $J^n\mathcal{G}$ such that $(\Gamma^{(n)})_0 = \pi^n_0 \circ \Gamma^{(n)} = \Gamma$.

Proposition 4.2. *If a jet groupoid $J^n\mathcal{G}$ admits a direct connection, then $J^n\mathcal{G}$ is the gauge groupoid of a jet prolonged principal bundle, that is,*

$$J^n\mathcal{G} \cong \mathcal{G}(W^nP) \tag{92}$$

for some principal bundle $P \to M$ such that $\mathcal{G} \cong \mathcal{G}(P)$.

Proof. By Proposition 2.4, we know that if $J^n\mathcal{G}$ admits a direct connection Σ, then $J^n\mathcal{G} \cong \mathcal{G}(Q)$ for some principal bundle $Q \to M$. Since the existence of Σ on $J^n\mathcal{G}$ implies the existence of the direct connection Σ_0 on $\mathcal{G}$, again by Proposition 2.4, $\mathcal{G}$ is the gauge groupoid of a principal bundle $P \to M$. By eq. (73) we then necessarily have $Q = W^nP$. $\square$

In the sequel we construct a jet prolongation

$$\mathrm{Pair}(M) \ni (y,x) \twoheadrightarrow \Gamma^{(n)}(y,x) \in (J^n\mathcal{G})^y_x$$

on the jet groupoid $J^n\mathcal{G}$ of a direct connection $\Gamma : \mathrm{Pair}(M) \twoheadrightarrow \mathcal{G}$ on a given groupoid $\mathcal{G}$.

4.2. **Exponential direct connection on the jet pair groupoid.** Assuming that the manifold M is endowed with an affine connection ∇^M, we first build an exponential bisection on $\mathrm{Pair}(M)$.

Definition 4.3. For any x in M, let U_x denote an open neighborhood of x chosen in such a way that any two points in U_x are linked by a unique geodesic. For any z in U_x, let $\exp_z : V_{0_z} \subset T_zM \to M$ denote the exponential map along geodesics and let $\tau^M(z,x) : T_xM \to T_zM$ be the parallel transport determined by ∇^M along the geodesic c^z_x linking x to z. For any choice of y in U_x, we call **exponential bisection from** x **to** y the local bisection, denoted by $\pi^y_x : z \mapsto (\varphi^y_x(z), z))$, defined on U_x by the diffeomorphism

$$\varphi^y_x : U_x \to M,\ z \mapsto \exp_z\big(\tau^M(z,x)\big(\underbrace{\exp_x^{-1}(y)}_{\in T_xM}\big)\big), \tag{93}$$

which clearly sends x to y.

The exponential bisection π^y_x which is smooth in (y,x), is the unique local bisection on $\mathrm{Pair}(M)$ with the following properties

(1) $\pi^y_x(x) = (y,x)$,
(2) if $y = x$ then $\pi^y_x(z) = (z,z)$ for any $z \in U_x$,
(3) the local vector field $U_x \ni z \mapsto \exp_z^{-1}\big(\varphi^y_x(z)\big) \in T_zM$ is parallel.

The last assertion follows from the fact the integral curve $[0,1] \ni t \mapsto \exp_x(tX)$ of any $X \in V_{0_x} \subset T_xM$ coincides with the geodesic c^y_x linking x to $y = \exp_x(X)$, then:

(4) for any z in U_x, $t \mapsto \varphi^{c^y_x(t)}_x(z)$ is the integral curve of $\tau^M_{c^y_x}X$ in T_zM, i.e. for any t we have

$$\varphi^{c^y_x(t)}_x(z) = \exp_z\big(\tau^M(z,x)(tX)\big) = \exp_z\big(t\,\tau^M(z,x)X\big), \tag{94}$$

(5) and in particular

$$\frac{d}{dt}\Big|_{t=0}\varphi^{c^y_x(t)}_x(z) = \tau^M(z,x)X. \tag{95}$$

We use the exponential bisection to construct a connection on $J^n\,\mathrm{Pair}(M)$.

Definition 4.4. Let $\mathcal{U} = \bigcup_{x\in M} U_x \times U_x$ be the diagonal domain covered by uniformly normal neighborhoods as in Definition 4.3. We call **exponential n-th order direct connection** on $\mathrm{Pair}(M)$ the local map $\Delta^{(n)}_M : \mathrm{Pair}(M) \twoheadrightarrow J^n\,\mathrm{Pair}(M)$ defined on any (y,x) in $\mathcal{U}$ by

$$\Delta^{(n)}_M(y,x) = j^n_x\big(z \mapsto \pi^y_x(z)\big). \tag{96}$$

Eq. (96) says that the bisection π_x^y is a "jet primitive" for the groupoid direct connection $\Delta_M^{(n)}$.

Note that $\Delta_M^{(n)}$ is a jet prolongation to $J^n\,\mathrm{Pair}(M)$ of the trivial connection Id on $\mathrm{Pair}(M)$. Also, by eq. (76) we have $J^1\,\mathrm{Pair}(M) \simeq \mathrm{Iso}(TM)$ from which it follows that $J^n\,\mathrm{Pair}(M) \simeq J^{n-1}\mathrm{Iso}(TM)$. We now show how $\Delta_M^{(n)}$ can be viewed as a jet prolongation of a direct connection on the frame groupoid $\mathcal{G}(FM) \simeq \mathrm{Iso}(TM)$ of M. The direct connection on $\mathrm{Iso}(TM)$ is built along the lines of Corollary 2.17 applied to the vector bundle $E = TM$ by means of the parallel transport along small geodesics induced by the affine connection ∇^M:

$$\Delta_M : \mathrm{Pair}(M) \twoheadrightarrow \mathrm{Iso}(TM) \tag{97}$$

given by $\Delta_M(y,x) = \left[\tau^{FM}(y,x)(\psi^x), \psi^x\right]$ for any choice of a frame ψ^x in F_xM, cf. §2.5.

Theorem 4.5. *The exponential n-th order direct connection $\Delta_M^{(n)}$ on $J^n\,\mathrm{Pair}(M)$ is a jet prolongation of the direct connection Δ_M defined in eq. (97).*

Proof. For given (y,x) in $\mathrm{Pair}(M)$, the exponential bisection $z \mapsto \pi_x^y(z) = (\varphi_x^y(z), z)$ is a smooth map defined in the neighborhood U_x of x, therefore its n-jet at x belongs to the jet groupoid $J^n\,\mathrm{Pair}(M)$ and lies in the fibre above $\pi_x^y(x) = (y,x)$. Since $\varphi_x^x(z) = \exp_z\left(\tau^M(z,x)\exp_x^{-1}(x)\right) = \exp_z(0) = z$, we have $j_x^n\varphi_x^x = j_x^n\mathrm{Id}_{U_x} = \mathrm{Id}_{U_x}(x) = x$ and therefore $\Delta_M^{(n)}(x,x) = u^n(x)$ is the unit 1_x in the jet groupoid $J^n\,\mathrm{Pair}(M)$. Hence the map $\Delta_M^{(n)}$ defined by eq. (96) is a direct connection on $J^n\,\mathrm{Pair}(M)$.

For $n = 1$, we have $\Delta_M^{(1)}(y,x) = j_x^1\pi_x^y = \left((y,x), d(\varphi_x^y)_x\right)$, where the differential at x of the diffeomorphism $z \mapsto \varphi_x^y(z)$ precisely gives the parallel transport $\tau^M(y,x)$ along the geodesic linking x to y and hence $\Delta_M^{(1)} = \Delta_M$. □

4.3. **Exponential infinitesimal connection on the jet pair groupoid.** We show that the infinitesimal connection induced by the direct connection $\Delta_M^{(n)}$ in Definition 4.4 coincides with the jet prolongation of the infinitesimal connection of Δ_M.

An affine connection ∇^M on M amounts to an infinitesimal connection

$$\delta_M : TM \to \mathrm{Der}(TM) \simeq \mathcal{L}(\mathrm{Iso}(TM))$$

on the frame groupoid $\mathrm{Iso}(TM)$. By eq. (86) we have $\mathrm{Der}(TM) \cong J^1TM$. According to [Mik07, §3] or [Kol09, §5], if ∇^M is torsion-free then δ_M can be prolonged to an n-th order Lie algebroid connection

$$\delta_M^{(n)} : TM \longrightarrow J^nTM \cong A(F^nM) \cong \mathcal{L}(J^n\,\mathrm{Pair}(M)), \tag{98}$$

called the **exponential n-th order prolongation of** δ_M. On a vector X in T_xM, it is defined as the n-jet

$$\delta_M^{(n)}(X) := j_{0_x}^n\left(D\exp_x(\tilde{X})\right), \tag{99}$$

where $D\exp_x : T(T_xM) \to TM$ is the differential of $\exp_x$ in a neighborhood of the null vector 0_x in T_xM and $\tilde{X}$ is the vector field on T_xM obtained by translation of X.

Note that $\delta_M^{(0)}(X) = D\exp_x(\tilde{X}) = \delta_M(X)$.

Theorem 4.6. *The infinitesimal connection $\delta^{\Delta_M^{(n)}}$ of the exponential n-th order direct connection $\Delta_M^{(n)}$ on $J^n\operatorname{Pair}(M)$ is the exponential n-th prolongation $\delta_M^{(n)}$ given in eq. (99) and we have*

$$\delta^{\Delta_M^{(n)}}(X) = \delta_M^{(n)}(X) = j_x^n\big(z \mapsto \tau^M(z,x)X\big) \tag{100}$$

for any X in T_xM.

Eq. (100) says that the parallel transport along small geodesics is a "jet primitive" for the Lie algebroid infinitesimal connection $\delta_M^{(n)}$.

Proof. Let us fix a point x in M and a vector X in T_xM, and show eq. (100). The jet in eq. (99) is computed for the function $T_xM \ni v \mapsto (D\exp_x)_v(\tilde{X}(v))$, where $(D\exp_x)_v : T_v(T_xM) \to T_{\exp_x(v)}M$ is defined on a vector Y_v in $T_v(T_xM)$ by the derivative

$$(D\exp_x)_v(Y_v) = \frac{d}{dt}\Big|_{t=0}\exp_x(v(t)).$$

Here, $t \mapsto v(t) \in T_xM$ is the integral curve for Y_v (i.e. such that $\frac{d}{dt}\big|_{t=0}v(t) = Y_v$) such that $v(0) = v$. Since T_xM is a vector space, for any v in T_xM there is a canonical isomorphism of vector spaces

$$T_xM \xrightarrow{\cong} T_{0_x}(T_xM) \xrightarrow{\cong} T_v(T_xM)$$

which identifies X in T_xM first to the vector $0_x + X$ in $T_{0_x}(T_xM)$ and then, by translation, to the vector $v + X$ in $T_v(T_xM)$. A generic vector Y_v in $T_v(T_xM)$ is therefore necessarily of the form $\tilde{X}(v) = v + X$ for some X in T_xM, and its integral curve through v is $v(t) = v + tX$. We have

$$\begin{aligned}(D\exp_x)_v(\tilde{X}(v)) &= \frac{d}{dt}\Big|_{t=0}\exp_x(v+tX)\\ &= \tau^M\big(\exp_x(v),x\big)X \in T_{\exp_x(v)}M,\end{aligned}$$

where c is the geodesic linking x to $\exp_x(v)$, and eq. (99) gives

$$\delta_M^{(n)}(X) = j_{0_x}^n\big(v \mapsto \tau^M\big(\exp_x(v),x\big)X\big). \tag{101}$$

On the other hand, setting $y = \exp_x(X)$ in U_x and denoting as before by $t \mapsto c_x^y(t) = \exp_x(tX)$ the geodesic linking x to y we find that

$$\begin{aligned}\delta^{\Delta_M^{(n)}}(X) &= \frac{d}{dt}\Big|_{t=0}\Delta_M^{(n)}(c_x^y(t),x) = \frac{d}{dt}\Big|_{t=0}j_x^n\big(z \mapsto \sigma_x^{c_x^y(t)}(z)\big)\\ &\equiv \frac{d}{dt}\Big|_{t=0}j_x^n\big(z \mapsto \varphi_x^{c_x^y(t)}(z)\big) \qquad \text{in } \mathcal{L}(J^n\operatorname{Pair}(M))_x\\ &= j_x^n\left(z \mapsto \frac{d}{dt}\Big|_{t=0}\varphi_x^{c_x^y(t)}(z)\right).\end{aligned} \tag{102}$$

We have used the fact that the derivatives in t and in the coordinates of z commute since all the maps are smooth and the variables t and z are mutually independent. Now, according to eq. (94), for any z in U_x the curve $t \mapsto \varphi_x^{c_x^y(t)}(z)$ is the integral curve of the vector $\tau^M(z,x)X \in T_zM$. From eq. (95) it then follows that

$$\delta^{\Delta_M^{(n)}}(X) = j_x^n\big(z \mapsto \tau^M(z,x)X\big).$$

Setting $z = \exp_x(v)$, we see that this formula and eq. (101) coincide, thus proving eq. (100). □

4.4. **Exponential direct connections on jet prolonged groupoids.** Let M be equipped with an affine connection ∇^M. In this paragraph we prove that any direct connection on a Lie groupoid $\mathcal{G} \rightrightarrows M$ can be prolonged to the jet groupoid $J^n\mathcal{G}$ using the exponential bisection π_x^y of $\mathrm{Pair}(M)$ given in Definition 4.3.

Definition 4.7. Let $\Gamma : \mathrm{Pair}(M) \ast\!\!\to \mathcal{G}$ be a direct connection defined on a diagonal domain $\mathcal{V} \subset M \times M$ and let $\mathcal{U} = \bigcup_{x \in M} U_x \times U_x$ be the diagonal domain covered by uniformly normal neighborhoods as in Definition 4.3.

We call **exponential n-th order prolongation of** Γ the local map $\Gamma^{(n)} : \mathrm{Pair}(M) \ast\!\!\to J^n\mathcal{G}$ defined by

$$\Gamma^{(n)}(y, x) = j_x^n \left(\Gamma \circ \pi_x^y\right) \in J^n\mathcal{G}_x^y \tag{103}$$

for any $(y, x) \in \mathcal{V} \cap \mathcal{U}$.

Corollary 4.8. *The exponential n-th prolongation $\Gamma^{(n)} : \mathrm{Pair}(M) \ast\!\!\to J^n\mathcal{G}$ of Γ indeed defines a direct connection on the jet groupoid which prolongs Γ. Moreover, it is compatible with the filtration on $J^n\mathcal{G}$*

$$\pi_{n-1}^n \Gamma^{(n)} = \Gamma^{(n-1)} \pi_{n-1}^n, \tag{104}$$

where $\pi_{n-1}^n : J^n\mathcal{G} \to J^{n-1}\mathcal{G}$ are the canonical projections and it factorises through $\Delta_M^{(n)}$, namely for any two points x, y in an exponential neighborhood of the diagonal of M, we have

$$\Gamma^{(n)}(y, x) = j_{(y,x)}^n \Gamma \circ \Delta_M^{(n)}(y, x). \tag{105}$$

Proof. We only need to check eq. (104). For a local bisection σ of $\mathcal{G}$ defined in a neighborhood of x, we have

$$\pi_{n-1}^n \left(\Gamma^{(n)}(y, x) j_x^n \sigma\right) = \pi_{n-1}^n \left(j_x^n(\Gamma \circ \pi_x^y)\right) = j_x^{n-1}(\Gamma \circ \pi_x^y) = \Gamma^{(n-1)}(y, x) j_x^{n-1}\sigma.$$

□

Remark 4.9. The family $\{\Gamma^{(n)}, n \in \mathbb{Z}_{\geqslant 0}\}$ is a projective system of direct connections in the sense of §6.4.

We now show that the infinitesimal connection $\delta^{\Gamma^{(n)}}$ of $\Gamma^{(n)}$ factorises through $\delta_M^{(n)} = j_x^n\big(z \mapsto \tau^M(z, x)\big)$ defined in eq. (99).

Proposition 4.10. *(1) The infinitesimal connection of $\Gamma^{(n)}$ can be expressed in terms of the infinitesimal connection of Γ as follows*

$$\delta^{\Gamma^{(n)}}(X) = j_x^n \left(z \mapsto \delta^\Gamma \left(\tau^M(z, x) X\right)\right), \tag{106}$$

for any X in T_xM. As expected, $\delta^{\Gamma^{(n)}}$ is therefore a jet prolongation of δ^Γ.

(2) Moreover, $\delta^{\Gamma^{(n)}}$ factorises through the exponential infinitesimal connection $\delta_M^{(n)} : TM \to J^nTM$ of eq. (98) as follows

$$\delta^{\Gamma^{(n)}} = J^n\delta^\Gamma \circ \delta_M^{(n)}, \tag{107}$$

where J^n is given by (70).

Proof.

(1) Let us fix a point x in M, a vector X in T_xM together with its integral curve $c_x^y(t) = \exp_x(tX)$ for t in $[0,1]$, where we have set $y := \exp_x(X)$. The infinitesimal connection of $\Gamma^{(n)}$ is computed using eq. (27) in §2.4. Since the derivatives in t commute with the partial derivatives with respect to the local coordinates in M and in TM, we have

$$\begin{aligned}\delta^{\Gamma^{(n)}}(X) &= \frac{d}{dt}\Big|_{t=0}\Gamma^{(n)}(c_x^y(t),x)\\ &= \frac{d}{dt}\Big|_{t=0}j_x^n\Big(z\mapsto \Gamma\circ\sigma_x^{c_x^y(t)}(z)\Big)\\ &= j_x^n\Big(z\mapsto \frac{d}{dt}\Big|_{t=0}\Gamma(\varphi_x^{c_x^y(t)}(z),z)\Big)\\ &= j_x^n\Big(z\mapsto \delta^\Gamma(\tau^M(z,x)X)\Big),\end{aligned} \tag{108}$$

where as before $\tau^M(z,x)$ is the parallel transport along the geodesic c_x^z linking x to z. The last equality follows from the fact that $\frac{d}{dt}\big|_{t=0}\varphi_x^{c_x^y(t)}(z) = \tau^M(z,x)X \in T_zM$ (cfr. eq. (95)). This proves eq. (106).

For $n=0$, eq. (106) says that $\delta^{\Gamma^{(0)}}(X) = \delta^\Gamma(\tau^M(x,x)X) = \delta^\Gamma(X)$, therefore $\delta^{\Gamma^{(n)}}$ is indeed a jet prolongation of δ^Γ.

(2) Let us now compute $j_x^n\delta^\Gamma\circ\delta_M^{(n)}(X)$ using the expression (101). Choosing a vector $v\in T_xM$, we set $z=\exp_x(v)$ and let as before c_x^z denote the geodesic from x to z. Then $t\mapsto \exp_x(v+tX)$ is the integral curve of the vector $\tau^M(\exp_x(v),x)X$, so that

$$\begin{aligned}j_x^n\delta^\Gamma\circ\delta_M^{(n)}(X) &= j_x^n\delta^\Gamma\Big(j_0^n\big(v\mapsto\tau^M(\exp_x(v),x)X\big)\Big)\\ &= j_0^n\Big(v\mapsto\delta^\Gamma\big(\exp_x(v+tX)\big)\Big)\\ &= j_0^n\Big(v\mapsto\frac{d}{dt}\Big|_{t=0}\Gamma(\exp_x(v+tX),x)\Big)\\ &= \frac{d}{dt}\Big|_{t=0}j_0^n\Big(v\mapsto\Gamma(\exp_x(v+tX),x)\Big)\\ &= \frac{d}{dt}\Big|_{t=0}j_x^n\Big(z\mapsto\Gamma(\exp_z(\tau^M(z,x)tX),x)\Big)\\ &= j_x^n\Big(z\mapsto\frac{d}{dt}\Big|_{t=0}\Gamma(\varphi_x^{c_x^y(t)}(z),x)\Big)\\ &= \delta^{\Gamma^{(n)}}(X)\end{aligned}$$

on the grounds of eq. (108). This proves eq. (107).

□

4.5. **The flat case.** Let $\mathcal{G}$ be a Lie groupoid. A flat infinitesimal connection

$$\delta: TM = \mathcal{L}(FM)\longrightarrow\mathcal{L}\mathcal{G}$$

i.e., a morphism of Lie algebroids, can be prolonged to a morphism of Lie algebroids

$$J^n\delta: J^nTM\simeq\mathcal{L}(F^nM)\to J^n\mathcal{L}(\mathcal{G}).$$

By Proposition 1.13, since $\mathcal{L}(J^n \operatorname{Pair}(M)) \simeq \mathcal{L}(F^n M) \simeq J^n TM$, the map $J^n\delta$ integrates to a (uniquely defined) local morphism

$$\Gamma^n_\delta : J^n \operatorname{Pair}(M) \circ\!\!\longrightarrow J^n\mathcal{G}. \tag{109}$$

If moreover M is equipped with a flat connection, the corresponding infinitesimal flat connection $TM \xrightarrow{\delta^{(n)}_M} \mathcal{L}(J^n(\operatorname{Pair}(M)))$ defined in eq. (98) canonically integrates to the flat direct connection $\Delta^{(n)}_M$.

The resulting composition

$$\Gamma^n_\delta \circ \Delta^{(n)}_M : \mathcal{P}(M) \circ\!\!\longrightarrow J^n\mathcal{G}$$

defines a direct connection on $J^n\mathcal{G}$, which is flat as a composition of local groupoid morphisms. The following theorem confirms via a straightforward algebraic argument, that this composition yields back the direct connection $\Gamma^{(n)}$ (cfr. eq. (105)). By similar algebraic arguments, in the flat case, $\Delta^{(n)}_M$ is shown to correspond to the direct connection induced by parallel transport on F^nM. We conjecture that this latter realisation only takes place if the underlying connection on M is flat.

Theorem 4.11. *Let M be a manifold equipped with a flat connection.*

(1) *The exponential direct connection $\Delta^{(n)}_M$ defined in eq. (96) is a flat direct connection induced by the parallel transport on F^nM along small geodesics determined by $\delta^{(n)}_M$ defined in eq. (99).*

(2) *Let $\widetilde{\Gamma}^n : \operatorname{Pair}(M) \circ\!\!\longrightarrow J^n\mathcal{G}$ be a flat direct connection on $J^n\mathcal{G}$, whose infinitesimal connection $\delta^{\widetilde{\Gamma}^n}$ factorises $\delta^{\widetilde{\Gamma}^n} = J^n\delta \circ \delta^{(n)}_M$ through the infinitesimal connection $\delta^{(n)}_M$ of eq. (99) by means of the n-th jet prolongation $J^n\delta$ of a flat infinitesimal connection δ on $\mathcal{G}$. In that case, $\widetilde{\Gamma}^n$ also factorises i.e.*

$$\widetilde{\Gamma}^n = \Gamma^n_\delta \circ \Delta^{(n)}_M, \tag{110}$$

with $\Delta^{(n)}_M$ as in eq. (96) and Γ^n_δ as in eq. (111) so that the corresponding diagramme commutes:

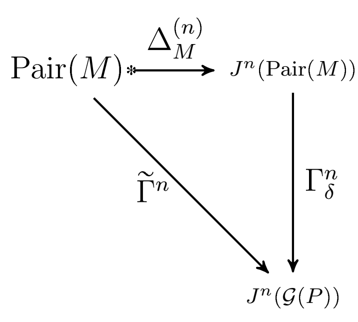


FIGURE 1. Connections on jet prolongations of gauge groupoids

(3) *Consequently, the direct connection $\Gamma^{(n)}$ on $J^n\mathcal{G}$ defined in eq. (103) factorises through $\Delta^{(n)}_M$ defined in eq. (96)*

$$\Gamma^{(n)} = \Gamma^n_{\delta^\Gamma} \circ \Delta^{(n)}_M, \tag{111}$$

where δ^Γ is the (flat) infinitesimal connection of Γ as defined in eq. (27).

The above identifications of direct connections hold on some exponential neighborhood of the diagonal of M.

Remark 4.12. Eq. (111) gives an alternative interpretation of eq. (105) when the direct connection is flat and the underlying manifold is equipped with a flat connection.

Proof. The proof uses Proposition 2.30, which says that a flat infinitesimal connection integrates to a unique direct connection modulo germ equivalence. It enables us to identify direct connections on some exponential neighborhood of the diagonal of M, identifications which we shall not specify in the proof.

(1) Formula (100) in Theorem 4.5 tells us that $\delta^{\Delta_M^{(n)}} = \delta_M^{(n)}$, which is flat as can be seen from its expression in terms of parallel transport of the underlying flat connection on M. Since a flat infinitesimal connection integrates to a unique direct connection modulo germ equivalence (cfr. Proposition 2.30), and $\delta_M^{(n)}$ is the infinitesimal connection of the direct connection on F^nM defined by the parallel transport along small geodesics determined by $\delta_M^{(n)}$, the statement follows.

(2) Since δ is flat, by eq. (111) its n- th jet prolongation $J^n\delta$ integrates to a flat connection Γ_δ^n. By Part (1) of the theorem, $\Delta_M^{(n)}$ is flat so that the composition $\Gamma_\delta^n \circ \Delta_M^{(n)}$ is flat. Eq. (110) then follows from the uniqueness (cfr. Proposition 2.30) of a flat direct connection with a given infinitesimal connection, here $\delta^{\tilde{\Gamma}^n}$.

(3) Eq. (111) follows from eq. (110) applied to the (flat) infinitesimal connection $\delta := \delta^\Gamma$.

□

4.6. **Direct connections on a jet frame groupoid from jet prolongations.** We consider the case where $\mathcal{G} = \mathrm{Iso}(E) \rightrightarrows M$ is the frame groupoid of a smooth vector bundle $\pi : E \to M$. Direct connections on the jet groupoid $J^n\mathrm{Iso}(E)$ are best described as linear operators acting on the jet bundle J^nE. For this, we first describe the inclusion $\rho : J^n\mathrm{Iso}(E) \hookrightarrow \mathrm{Iso}(J^nE)$.

An element in $J^n\mathrm{Iso}(E)$ is the jet $j^n_{x_0}\sigma$ of a local bisection $\sigma : U_{x_0} \to \mathrm{Iso}(E)$ defined on an open neighborhood U_{x_0} of a point x_0 in M, that is, a smooth map such that $\sigma(x)$ in $\mathrm{Iso}(E)^y_x$ for any x in U_{x_0}, where $y = \varphi_\sigma(x)$ is the image of x by the diffeomorphism $\varphi_\sigma = t \circ \sigma$ of M defined on U_{x_0} associated to σ. This means that, for any x in U_{x_0}, $\sigma(x) : E_x \to E_y$ is a linear isomorphism between fibres of E above φ_σ-related points.

The image $\rho(j^n_{x_0}\sigma)$ in $\mathrm{Iso}(J^nE)$ is a linear isomorphism $\rho(j^n_{x_0}\sigma) : J^n_{x_0}E \to J^n_{y_0}E$ between fibres of J^nE above φ_σ-related points, where we set $y_0 = \varphi_\sigma(x_0)$, defined on the jet $j^n_{x_0}f \in J^n_{x_0}E$ of a smooth local section f of E around x_0 as

$$\begin{aligned} \rho(j^n_{x_0}\sigma)(j^n_{x_0}f) &:= j^n_{y_0}\Big(y \mapsto \sigma\big(\varphi_\sigma^{-1}(y)\big) \cdot f\big(\varphi_\sigma^{-1}(y)\big)\Big) \\ &= j^n_{x_0}\Big(x \mapsto \sigma(x) \cdot f(x)\Big) \circ j^n_{y_0}\varphi_\sigma^{-1}, \end{aligned} \tag{112}$$

where φ_σ^{-1} is the inverse diffeomorphism of φ_σ and where we denote by $\cdot$ the linear action of $\sigma(x)$ on $f(x) \in E_x$, as in eq. (**??**), and by $\circ$ the jet composition as in eq. (43). The map ρ will be usually omitted, and the linear action defined by eq. (112) will be simply denoted by $j^n_{x_0}\sigma \cdot j^n_{x_0}f$.

Assume that the manifold M is equipped with an affine connection ∇^M, and let $\Gamma : \mathrm{Pair}(M) \ast\!\!\to \mathrm{Iso}(E)$ be a direct connection on the frame groupoid, defined on a diagonal domain $\mathcal{U}$ in $\mathrm{Pair}(M)$. Then, for any (y_0, x_0) in $\mathcal{U}$, $\Gamma(y_0, x_0) : E_{x_0} \to E_{y_0}$ is a linear isomorphism. As before, we denote by $\cdot$ the action of $\Gamma(y_0, x_0)$ on any element e_{x_0} in E_{x_0}, that is, we write

$$E_{x_0} \ni e_{x_0} \longmapsto \Gamma(y_0, x_0) \cdot e_{x_0} \in E_{y_0}.$$

From Γ we can build the exponential direct connection on $J^n\mathrm{Iso}(E)$ as in eq. (103). Namely, for $(y_0, x_0) \in \mathcal{U}$, we consider the exponential bisection $\pi_{x_0}^{y_0}$ and its associated diffeomorphism $\varphi_{x_0}^{y_0}$ given in Definition 4.3. Then, according to eq. (112), $\Gamma^{(n)}(y_0, x_0)$ is the linear isomorphism $J^n_{x_0}E \to J^n_{y_0}E$ acting as

$$\begin{aligned}\Gamma^{(n)}(y_0, x_0) \cdot j^n_{x_0} f &= j^n_{y_0}\left(y \mapsto \Gamma(y, \psi_{y_0}^{x_0}(y)) \cdot f(\psi_{y_0}^{x_0}(y))\right) \\ &= j^n_{x_0}\left(x \mapsto \Gamma(\varphi_{x_0}^{y_0}(x), x) \cdot f(x)\right) \circ j^n_{y_0}\psi_{y_0}^{x_0},\end{aligned}$$

where f is a local section of E around x_0 and $\psi_{y_0}^{x_0}$ is the inverse diffeomorphism of $\varphi_{x_0}^{y_0}$.

This jet prolongation is compatible with the filtration on J^nE, i.e.

$$\pi^n_{n-1}\Gamma^{(n)} = \Gamma^{(n-1)}\pi^n_{n-1}, \tag{113}$$

where $\pi^n_{n-1} : J^nE \to J^{n-1}E$ was defined in eq. (51). Indeed,

$$\begin{aligned}\pi^n_{n-1}\left(\Gamma^{(n)}(y_0, x_0) \cdot j^n_{x_0} f\right) &= \pi^n_{n-1}\left(j^n_{y_0}\left(y \mapsto \Gamma(y, \psi_{y_0}^{x_0}(y)) \cdot f(\psi_{y_0}^{x_0}(y))\right)\right) \\ &= j^{n-1}_{y_0}\left(y \mapsto \Gamma(y, \psi_{y_0}^{x_0}(y)) \cdot f(\psi_{y_0}^{x_0}(y))\right) \\ &= \Gamma^{(n-1)}(y_0, x_0) \cdot j^{n-1}_{x_0} f \\ &= \Gamma^{(n-1)}(y_0, x_0) \cdot \pi^n_{n-1}\left(j^n_{x_0} f\right).\end{aligned}$$

It follows that the family $\{\Gamma^{(n)}, n \in \mathbb{Z}_{\geqslant 0}\}$ is a projective system of direct connections in the sense of §6.4.

Example 4.13. We take $E = M \times \mathbb{R}^r$ with $M = \mathbb{R}^d$ and let $\mathbf{1} : x \mapsto \mathbf{1}_x = (x, \underbrace{(1, \cdots, 1)}_{r \text{ times}})$ be a given constant section. We equip $\pi : E \to M$ with the trivial direct connection Γ on E defined by $\Gamma(y, x) \cdot \mathbf{1}_x = \mathbf{1}_y$. Then, for a function $f : M \to \mathbb{R}$, viewed at a point x in M as an element $(x, f(x)) = f(x)\,\mathbf{1}_x$ of the fibre E_x, we have $\Gamma(y, x) \cdot (x, f(x)) = (y, f(x))$. It follows that

$$\begin{aligned}\Gamma^{(n)}(y_0, x_0) \cdot j^n_{x_0} f\,\mathbf{1_{x_0}} &= j^n_{y_0}\left(y \mapsto \Gamma(y, \psi_{y_0}^{x_0}(y)) \cdot f(\psi_{y_0}^{x_0}(y))\,\mathbf{1}_y\right) \\ &= j^n_{y_0}\left(y \mapsto f(\psi_{y_0}^{x_0}(y))\,\mathbf{1}_{\psi_{y_0}^{x_0}(y)}\right) \\ &= \left(j^n_{x_0} f \circ j^n_{y_0}\psi_{y_0}^{x_0}\right)\mathbf{1}_{y_0} \in J^n_{y_0}E.\end{aligned}$$

When M is the space $\mathbb{R}^d$ equipped with the trivial connection ∇^M given by the Levi-Civita connection for the canonical metric on $\mathbb{R}^n$, then $\psi_{y_0}^{x_0}(y) = y + x_0 - y_0$ and the above formula boils down to:

$$\Gamma^{(n)}(y_0, x_0) \cdot j^n_{x_0} f\,\mathbf{1}_{x_0} = j^n_{x_0} f\,\mathbf{1}_{y_0}. \tag{114}$$

4.7. **Direct connections on a frame groupoid from Taylor expansions.** Direct connections on the frame groupoid $\mathrm{Iso}(J^nE)$ of a jet bundle are of special interest in the context of regularity structures. This is the gauge groupoid of the frame bundle $F(J^nE)$ with structure group $GL(P^n_{d,r})$, but it is *not* the jet prolongation of a groupoid, because the group $GL(P^n_{d,r})$ is not the jet prolongation of a structure group and the frame bundle $F(J^nE)$ is not the jet prolongation of a principal bundle, cf. eq. (81) and eq. (77) in Example 3.4.

Until now we have focussed our attention on direct connections on *jet groupoids of frame bundles* rather than on *frame groupoids of jet bundles*. This paragraph is dedicated to the latter. By means of local Taylor expansions, we build a direct connection on $\mathrm{Iso}(J^nE)$, which is *not a jet prolongation* of Γ.

As in the previous section, the manifold M is equipped with an affine connection ∇^M, and $\Gamma : \mathrm{Pair}(M) \twoheadrightarrow \mathrm{Iso}(E)$ is a direct connection on the frame groupoid with infinitesimal connection $\nabla := \nabla^\Gamma$, see eq. (28) (for convenience, we drop the superscript Γ).

Taylor expansions.

Let us first assign to an element $j^n_{x_0}(f)$ in $J^n_{x_0}E$, the Taylor local expansion at x_0

$$\Pi^\Gamma_{x_0}(j^n_{x_0}f)(x) = \Gamma(x,x_0)\sum_{k=0}^n \frac{1}{k!}\frac{d^k}{dt^k}\left(\Gamma^{-1}(x_0,c_{x_0}(t))(f(c_{x_0}(t)))\right)\Big|_{t=0} \in E_x, \tag{115}$$

where as before, $c_{x_0}(t)$ is the unique geodesic curve linking x_0 to $x = c_{x_0}(1)$.

This compares with the local Taylor expansion of [DDD19, Definition 76] which uses higher connections defined as follows. The linear derivation ∇ on E extends to a linear connection $\nabla^{(n)}$ on $(T^*M)^{\otimes n} \otimes E$ for any $n \in \mathbb{Z}_{\geqslant 0}$, defined for $\alpha_1, \cdots, \alpha_n$ in $C^\infty(M, T^*M)$ and f in $C^\infty(M,E)$ by

$$\begin{aligned}&\nabla^{(n)}_X(\alpha_1\otimes\cdots\otimes\alpha_n\otimes f)\\ =\;&\sum_{j=1}^n \alpha_1\otimes\cdots\otimes\nabla^M_X\alpha_j\otimes\cdots\otimes\alpha_n\otimes f + \alpha_1\otimes\cdots\cdots\otimes\alpha_n\otimes\nabla_X f.\end{aligned} \tag{116}$$

For any smooth section f of E, we set

$$\nabla^n f := \nabla^{(n)}\left(\nabla^{(n-1)}\left(\cdots\nabla^{(1)}(\nabla f)\cdots\right)\right) \in C^\infty(M,(T^*M)^{\otimes n+1}\otimes E) \tag{117}$$

and [JL14, §2.1] (compare with [DDD19, eq.(27)] in the case $E = M\times\mathbb{R}$)

$$\mathrm{Sym}^n[\nabla^n f] := [\mathrm{Sym}^n\otimes \mathrm{id}_E](\nabla^n f). \tag{118}$$

Here (with a slight abuse of notation) on the r.h.s. we use the symmetrising map $\mathrm{Sym}^n : T^*M^{\otimes n} \to S^n(T^*M)$ defined by $\mathrm{Sym}^n(\alpha_1\otimes\cdots\otimes\alpha_n) := \frac{1}{n!}\sum_{\sigma\in\Sigma_n}\alpha_{\sigma(1)}\otimes\cdots\otimes\alpha_{\sigma(n)}$. Note that $\mathrm{Sym}^n[\nabla^n f](X^n) = \nabla^n f(X,\cdots,X)$ for any X in T_xM at an arbitrary point x in M.

For a smooth local section g of E around $x \in M$, we further set

$$D^k(g)(X^k) := \frac{d^k}{dt^k}\left(g\left(c_{x_0}(t)\right)\right)\Big|_{t=0} \quad \forall k \in \mathbf{N}. \tag{119}$$

Lemma 4.14. *The Taylor expansion eq. (115) reads*

$$\begin{aligned}\Pi^\Gamma_{x_0}(j^n_{x_0}f)(x) &= \Gamma(x,x_0)\sum_{k=0}^n\frac{1}{k!}D^k\left(z\mapsto\Gamma^{-1}(x_0,z)(f(z))\right)\left(\exp^{-1}_{x_0}x\right)^{\otimes k}\\ &= \Gamma(x,x_0)\sum_{k=0}^n\frac{1}{k!}\nabla^k f\left(\exp^{-1}_{x_0}x\right)^{\otimes k},\end{aligned} \tag{120}$$

with $\Gamma^{-1}(x,y) := \Gamma(y,x)^{-1}$.

Proof. The first identity follows from eq. (119) and the second identity easily follows from the composition rule for differentiation combined with the fact that $\nabla^M_{\dot{c}_{x_0}}\dot{c}_{x_0} = 0$ (compare with [JL14, 2.2 Lemma]). Indeed, for any small enough positive t and any k in $\mathbf{N}$, we have

$$\begin{aligned}\Gamma(c_{x_0}(t), x_0)\, \frac{d^k}{dt^k}\left(\Gamma^{-1}(x_0, c_{x_0}(t)) f(c_{x_0}(t))\right) &= \nabla^k_{c_{x_0}(t)} f(\dot c_{x_0}(t), \cdots, \dot c_{x_0}(t)) \\ &= D^k\left(\Gamma^{-1}(x_0, \cdot)(f)\right)(\dot c_{x_0}(t), \cdots, \dot c_{x_0}(t)),\end{aligned} \tag{121}$$

which at $t = 0$ and setting $X := \dot c_{x_0}(0) \in T_{x_0}M$, gives the following generalisation of eq. (28)

$$\frac{d^n}{dt^n}\left(\Gamma^{-1}(x_0, c_{x_0}(t))\, f(c_{x_0}(t))\right)|_{t=0} = \nabla^n f(X^n) = D^n\left(\Gamma^{-1}(x_0, \cdot)(f)\right)(X^n) \quad \forall n \in \mathbf{N}. \tag{122}$$

Inserting eq. (122) in the Taylor expansion (115) yields eq. (120). □

Example 4.15. As in Example 4.13, we take $E = M \times \mathbb{R}^r$ and $M = \mathbb{R}^d$. We equip the frame groupoid $\mathrm{Iso}(E)$ with the trivial direct connection $\Gamma(y, x)\, f(x)\, \mathbf{1}_x = f(x)\, \mathbf{1}_y$, where $\mathbf{1}_x = (x, (1, \cdots, 1))$ is the global constant section trivial linear connection. In that case, $\nabla^\Gamma f := df$ so that the Taylor expansion reads

$$\begin{aligned}\Pi^\Gamma_{x_0}\underbrace{(j^n_{x_0} f\, \mathbf{1}_{x_0})}_{\in J^n_{x_0}E}(x) &= \sum_{|\alpha| \leqslant n} \frac{1}{\alpha!}\left(\Gamma(x, x_0)\, \partial^\alpha f(x_0)\, \mathbf{1}_{x_0}\right)(x - x_0)^\alpha \\ &= \left(\sum_{|\alpha| \leqslant n} \frac{1}{\alpha!}\partial^\alpha f(x_0)\,(x - x_0)^\alpha\right)\mathbf{1}_x \in E_x \simeq \{x\} \times \mathbb{R}^r.\end{aligned}$$

Inspired by [DDD19, Definition 80], we set the following definition.

Definition 4.16. For any pair (x_0, y_0) in $\mathcal{U} \subset \mathrm{Pair}(M)$, and any local section f of E in a small neighborhood of x_0, we define $\widetilde\Gamma^{(n)}(y_0, x_0)$ in $E^*_{x_0} \otimes E_{y_0}$ by

$$\widetilde\Gamma^{(n)}(y_0, x_0) \cdot j^n_{x_0} f := j^n_{y_0}\left(y \mapsto \Pi^\Gamma_{x_0}(j^n_{x_0} f)(y)\right). \tag{123}$$

Example 4.17. We consider the same setting as in Example 4.15, namely the trivial direct connection Γ on $\mathrm{Iso}(E)$ with $E = M \times \mathbb{R}^r$ the trivial vector bundle over $M = \mathbb{R}^d$. In order to compute $\widetilde\Gamma^{(n)}$ we first compute

$$\partial^\beta(\bullet - x_0)^\alpha|_{y_0} = \frac{\alpha!}{(\alpha - \beta)!}(y_0 - x_0)^{\alpha - \beta} \quad \text{for} \quad \beta \leqslant \alpha \quad \text{and} \quad \partial^\beta(\bullet - x_0)^\alpha|_{y_0} = 0 \quad \text{for} \quad \beta > \alpha,$$

where $\beta \leqslant \alpha$ stands for $\alpha_i \leqslant \beta_i$ for any $i \in [[1, d]]$ and $\alpha > \beta$ when this does not hold. Viewing an n-th jet as an n-th Taylor polynomial $j^n_{x_0} f = \sum_{|\alpha| \leqslant n} \frac{\partial^\alpha(x_0) f}{\alpha!} X^\alpha$ with $X^\alpha :=$

$X_1^{\alpha_1} \cdots X_d^{\alpha_d}$, we have

$$\begin{aligned}
&\widetilde{\Gamma}^{(n)}(y_0, x_0) \cdot \left(j^n_{x_0} f \,\mathbf{1}_{x_0}\right) \\
=\;& j^n_{y_0}\left(y \mapsto \sum_{|\alpha| \leqslant n} \frac{1}{\alpha!} \partial^\alpha f(x_0)\,(y - x_0)^\alpha\right) \mathbf{1}_{y_0} \\
=\;& \sum_{|\alpha| \leqslant n} \frac{\partial^\alpha f(x_0)}{\alpha!}\, j^n_{y_0}(\bullet - x_0)^\alpha \,\mathbf{1}_{y_0} \\
=\;& \sum_{|\alpha| \leqslant n} \frac{\partial^\alpha f(x_0)}{\alpha!} \left(\sum_{|\beta| \leqslant n} \frac{\partial^\beta (\bullet - x_0)^\alpha|_{y_0}}{\beta!}\, X^\beta\right) \mathbf{1}_{y_0} \\
=\;& \sum_{|\alpha| \leqslant n} \left[\sum_{\beta \leqslant \alpha} \frac{\partial^\alpha f(x_0)}{\beta!\,(\alpha - \beta)!}\,(y_0 - x_0)^{\alpha - \beta}\, X^\beta\right] \mathbf{1}_{y_0} \qquad (124) \\
=\;& \underbrace{\left(\sum_{|\alpha| \leqslant n} \frac{\partial^\alpha f(x_0)}{\alpha!}\,(y_0 - x_0)^\alpha\, X^\alpha\right)}_{\beta = \alpha} \mathbf{1}_{y_0} + \sum_{|\alpha| \leqslant n} \left[\sum_{\beta < \alpha} \frac{\partial^\alpha f(x_0)}{\beta!\,(\alpha - \beta)!}\,(y_0 - x_0)^{\alpha - \beta}\, X^\beta\right] \mathbf{1}_{y_0} \\
=\;& j^n_{x_0} f \,\mathbf{1}_{y_0} + \sum_{|\alpha| \leqslant n} \left[\sum_{\beta < \alpha} \frac{\partial^\alpha f(x_0)}{\beta!\,(\alpha - \beta)!}\,(y_0 - x_0)^{\alpha - \beta}\, X^\beta\right] \mathbf{1}_{y_0} \\
=\;& \Gamma^{(n)}(y_0, x_0) \cdot \left(j^n_{x_0} f \,\mathbf{1}_{x_0}\right) + \sum_{|\alpha| \leqslant n} \left[\sum_{\beta < \alpha} \frac{\partial^\alpha f(x_0)}{\beta!\,(\alpha - \beta)!}\,(y_0 - x_0)^{\alpha - \beta}\, X^\beta\right] \mathbf{1}_{y_0}.
\end{aligned}$$

In dimension 1 this reads:

$$\begin{aligned}
&\widetilde{\Gamma}^{(n)}(y_0, x_0) \cdot \left(j^n_{x_0} f \,\mathbf{1}_{x_0}\right) \\
=\;& \begin{bmatrix}
1 & y_0 - x_0 & (y_0 - x_0)^2 & \cdots & \cdots & (y_0 - x_0)^n \\
0 & 1 & 2\,(y_0 - x_0) & \cdots & \cdots & n\,(y_0 - x_0)^{n-1} \\
0 & 0 & 1 & 3\,(y_0 - x_0) & \cdots & (n-1)\,(y_0 - x_0)^{n-2} \\
0 & 0 & 0 & 1 \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & & \cdots & n(y_0 - x_0) \\
\cdots & \cdots & \cdots & \cdots & 0 & 1
\end{bmatrix}
\begin{bmatrix}
f(x_0) \\ f'(x_0) \\ \cdots \\ \cdots \\ \cdots \\ \frac{f^{(n-1)}(x_0)}{(n-1)!} \\ \frac{f^{(n)}(x_0)}{n!}
\end{bmatrix} \cdot \mathbf{1}_{x_0}
\end{aligned}$$

These computations generalise to sections of a vector bundle $E \to M$ using the following identification via the following maps (here $n \in \mathbf{N}$) [Pa65, Chapter IV §9, Corollary of Theorem 7], [JL14, Lemma 2.1]

$$\begin{aligned}
S^n_{\nabla, \nabla^M} : J^n E &\longrightarrow \oplus_{k=0}^n S^k(T^*M) \otimes E \\
j^n_x f &\longmapsto \left(f(x), \nabla f(x), \cdots, \mathrm{Sym}^k[\nabla^k f](x), \cdots, \mathrm{Sym}^n[\nabla^n f](x)\right). \qquad (125)
\end{aligned}$$

In a polynomial representation, this reads

$$j^n_x g = \sum_{j=0}^{n} \mathrm{Sym}^j[\nabla^j_x g]\, X^j, \tag{126}$$

where $\nabla = \nabla^\Gamma$ is the infinitesimal connection induced by Γ. With these conventions,

$$\Gamma^{(n)}(y_0, x_0)(j^n_{x_0} g) = j^n_{y_0}(\Gamma(\bullet, x_0)\, g(x_0)) = \sum_{j=0}^{n} \mathrm{Sym}^j[\nabla^j_{y_0}(\Gamma(\bullet, x_0)\, g(x_0))]\, X^j. \tag{127}$$

Proposition 4.18. *$\widetilde{\Gamma}^{(n)}$ defines a parallelism on J^nE in the sense of eq. (21), and hence a direct connection $\widetilde{\Gamma}^{(n)} : \mathrm{Pair}(M) \twoheadrightarrow \mathrm{Iso}(J^nE)$ on the frame groupoid of J^nE (which we denote by the same symbol). It compares with $\Gamma^{(n)}$ as follows*

$$\begin{aligned} \widetilde{\Gamma}^{(n)}(y_0, x_0)\left(j^n_{x_0} f\right) &= \Gamma^{(n)}(y_0, x_0)\,(j^n_{y_0} f) \\ &+ \Gamma^{(n-1)}(y_0, x_0)\left(\sum_{|\alpha|\leqslant n}\left[\sum_{\beta<\alpha} \frac{\mathrm{Sym}[\nabla^\alpha_{x_0}](f)(\exp^{-1}_{x_0}(y_0))^{\alpha-\beta}}{\beta!\,(\alpha-\beta)!} X^\beta\right]\right). \end{aligned} \tag{128}$$

Proof.

- To show that $\widetilde{\Gamma}^{(n)}$ defines a direct connection, by Proposition 2.6, all we need to prove is that $\widetilde{\Gamma}^{(n)}(x_0, x_0) = \mathrm{Id}_{J^n_{x_0}E}$. So we need to check that $j^n_{x_0}\left(\Pi^\Gamma_{x_0}(j^n_{x_0} f)\right) = j^n_{x_0}(f)$. In order to compute $j^n_{x_0}\left(\Pi^\Gamma(j^n_{x_0} f)\right)$ we use the local identification via the maps (125).

  By definition of $\Pi^\Gamma_{x_0}$

$$\Pi^\Gamma_{x_0}(j^n_{x_0} f)(x) = \Gamma(x, x_0) \sum_{k=0}^{n} \frac{1}{k!}\frac{d^k}{dt^k}\left(\Gamma^{-1}(x_0, c_{x_0}(t))(f(c_{x_0}(t)))\right)\Big|_{t=0}$$

  and by eq. (122), applied to $V = \dot{c}_{x_0}(0)$ in $T_{x_0}M$

$$\nabla^k_{x_0} f(V^k) = \frac{d^k}{dt^k}\left(\Gamma^{-1}(x_0, c_{x_0}(t))\, f(c_{x_0}(t))\right)|_{t=0},$$

  we have

$$\begin{aligned} &\nabla^k_{x_0}\left(x \mapsto \Pi^\Gamma(j^n_{x_0} f)\right)(V^k) \\ =\ &\nabla^k_{x_0}\left(x \mapsto \Gamma(x, x_0) \sum_{j=0}^{n} \frac{\nabla^j f}{j!}(\exp^{-1}_{x_0} x)^{\otimes j}\right)(V^k) \\ =\ &\frac{d^k}{dt^k}\left(\sum_{j=0}^{n} \frac{d^j}{dt^j}\left(\Gamma^{-1}(x_0, c_{x_0}(t))\, f(c_{x_0}(t))\right)\frac{(tV)^{\otimes j}}{j!}\right)_{|t=0} \\ =\ &\sum_{j=0}^{k} \frac{d^k}{dt^k}\left(\Gamma^{-1}(x_0, c_{x_0}(t))\, f(c_{x_0}(t))\right)_{|t=0}(V^k) \\ &\text{since}\quad \left(\frac{d^k}{dt^k}(tV)^{\otimes j}\right)_{|t=0} = \delta_{k-j}\, k!\, V^{\otimes k} \\ =\ &\left(\nabla^k_{x_0} f\right)(V^k), \end{aligned}$$

where we have used the definition of $\nabla^k_{x_0}$ in eq. (116).

- To prove eq. (128), we closely follow the computation in eq. (124). As before, we set $V := \dot{c}_{x_0}(0)$ so that $y_0 := c_{x_0}(1) = \exp_{x_0}(V)$, and

$$\nabla^\beta_{x_0} \exp^{-1}_{x_0}(\bullet)^\alpha = \frac{\alpha!}{(\alpha-\beta)!} V^{\alpha-\beta} \quad \text{for} \quad \beta \leqslant \alpha \quad \text{and} \quad \nabla^\beta_{x_0} \exp^{-1}_{x_0}(\bullet)^\alpha = 0 \qquad \text{for} \quad \beta > \alpha.$$

Using the description of jets given by eq. (125) and the Taylor expansion eq. (120), we write

$$\begin{aligned}
&\widetilde{\Gamma}^{(n)}(y_0, x_0) \cdot \left(j^n_{x_0} f\right) \\
= \quad & j^n_{y_0} \left(x \mapsto \Gamma(x, x_0) \underbrace{\sum_{|\alpha| \leqslant n} \frac{\mathrm{Sym}[\nabla^\alpha_{x_0} f]}{\alpha!} (\exp^{-1}_{x_0} x)^{\otimes \alpha}}_{g(x_0) \in E_{x_0}} \right) \\
\underset{\text{eq. (127)}}{=} \quad & \Gamma^{(n)}(y_0, x_0)\, j^n_{x_0} \left(x \mapsto \sum_{|\alpha| \leqslant n} \frac{\mathrm{Sym}[\nabla^\alpha_{x_0} f]}{\alpha!} (\exp^{-1}_{x_0} x)^{\otimes \alpha} \right). \\
= \quad & \Gamma^{(n)}(y_0, x_0) \sum_{|\alpha| \leqslant n} \sum_{|\beta| \leqslant n} \frac{\mathrm{Sym}[\nabla^\alpha_{x_0} f](V^{\alpha-\beta})}{\beta!\,(\alpha-\beta)!} X^\beta \\
= \quad & \Gamma^{(n)}(y_0, x_0) \underbrace{\sum_{|\alpha| \leqslant n} \frac{\mathrm{Sym}[\nabla^\alpha_{x_0} f]}{\alpha!} X^\alpha}_{\beta = \alpha} \\
+ \quad & \Gamma^{(n-1)}(y_0, x_0) \sum_{|\alpha| \leqslant n} \left[\sum_{\beta < \alpha} \frac{\mathrm{Sym}[\nabla^\alpha_{x_0} f](V^{\alpha-\beta})}{\beta!\,(\alpha-\beta)!} X^\beta \right] \\
= \quad & \Gamma^{(n)}(y_0, x_0)(j^n_{x_0} f) + \Gamma^{(n-1)}(y_0, x_0) \left(\sum_{|\alpha| \leqslant n} \left[\sum_{\beta < \alpha} \frac{\mathrm{Sym}[\nabla^\alpha_{x_0} f](\exp^{-1}_{x_0}(y_0))^{\alpha-\beta}}{\beta!\,(\alpha-\beta)!} X^\beta \right] \right)
\end{aligned}$$

□

The following observation stresses the difference in nature between the connections $\widetilde{\Gamma}^{(n)}$ and $\Gamma^{(n)}$, which we recall gives rise to a direct connection on $\mathrm{Iso}(J^\infty E)$.

Remark 4.19.
- We have

$$\begin{aligned}
\pi^n_0 \left(\widetilde{\Gamma}^{(n)}(y_0, x_0) \cdot j^n_{x_0} f \right) &= j^0_{y_0} \left(x \mapsto \Gamma(x, x_0) \sum_{k=0}^n \frac{1}{k!} \mathrm{Sym}^k[\nabla^k f](x_0) \, \left(\exp^{-1}_{x_0} x\right)^{\otimes k} \right) \\
&= \Gamma(y_0, x_0) \sum_{k=0}^n \frac{1}{k!} \mathrm{Sym}^k[\nabla^k f](x_0) \, \left(\exp^{-1}_{x_0} y_0\right)^{\otimes k} \\
&= \Gamma(y_0, x_0)\, f(x_0) + \sum_{k=1}^n \frac{1}{k!} \mathrm{Sym}^k[\nabla^k f](x_0) \, \left(\exp^{-1}_{x_0} y_0\right)^{\otimes k} \\
&\neq \Gamma(y_0, x_0)\, f(x_0),
\end{aligned}$$

which shows that $\tilde{\Gamma}^{(n)}$ *does not prolong* Γ.

- This further shows that the connection $\widetilde{\Gamma}^{(n)}$ does not satisfy condition (113) with Γ replaced by $\widetilde{\Gamma}$. Thus the family $\left(\widetilde{\Gamma}^{(n)}, n \in \mathbb{Z}_{\geqslant 0}\right)$ *does not induce a direct connection on* $\operatorname{Iso}(J^\infty E)$.

5. Geometric pre-regularity structures for a vector bundle

In this section we discuss the geometric framework underlying regularity structures on sections of a vector bundle of finite rank on a manifold M by means of a direct connection. We define a geometric pre-regularity structure which keeps track of the structure group in the form of a groupoid and its action on the vector bundle. It is described in the projective setup adapted to the grading underlying regularity structures and inherent to perturbative approaches to quantum field theory. The geometric pre-regularity structure comes with a geometric pre-model which encompasses the geometric data required for a full fledged model as defined in the context of regularity structures, leaving out the analytic requirements, hence the prefix "pre" in front of "regularity structures".

We revisit Dahlqvist, Diehl and Driver's [DDD19] polynomial regularity structures in the language of geometric polynomial structures.

5.1. **The abstract setup.** We work in a projective set up and refer the reader to Appendix 6 for the relevant notations.

Definition 5.1. Let M be a smooth manifold endowed with a connection ∇^M (not necessarily torsion-free) on TM with positive injectivity radius r_M.

We call **geometric pre-regularity structure** on M the data $(A, E, \mathcal{G}, \rho)$ where:

(1) $A \subset \mathbb{R}$ is a discrete set of indices, that we shall call homogeneities following [H14], directed by the order relation in $\mathbb{R}$, with no accumulation point and bounded from below;

(2) $E = \varprojlim_{\alpha \in A} E_\alpha$ is a projective limit of vector bundles $E_\alpha \to M$ as in (145), called the **model bundle**;

(3) $\mathcal{G} = \varprojlim_{\alpha \in A} \mathcal{G}_\alpha \rightrightarrows M$ is a prounipotent gauge groupoid as in (154), called the **structure Lie groupoid**;

(4) $\iota : \mathcal{G} \hookrightarrow \varprojlim_{\alpha \in A} \mathrm{Iso}(E_\alpha) \subset \mathrm{Iso}(E)$ is an injective morphism of prounipotent groupoids,called the **$\mathcal{G}$-structure on** E, consisting of a family $\iota_\alpha : \mathcal{G}_\alpha \hookrightarrow \mathrm{Iso}(E_\alpha), \alpha \in A$ of injective morphisms of groupoids. It is equivalently presented as a faithful linear representation

$$\rho : \mathcal{G} \times_M E \longrightarrow E, \ (g_y^x, a_x) \mapsto \rho(g_y^x)(a_x) \tag{129}$$

which preserves the projective systems of $\mathcal{G}$ and E.

Given a real vector bundle $E_0 \to M$ of rank r, we call **geometric pre-model** for $(A, E, \mathcal{G}, \rho)$ on E_0 the data (Π, Γ):

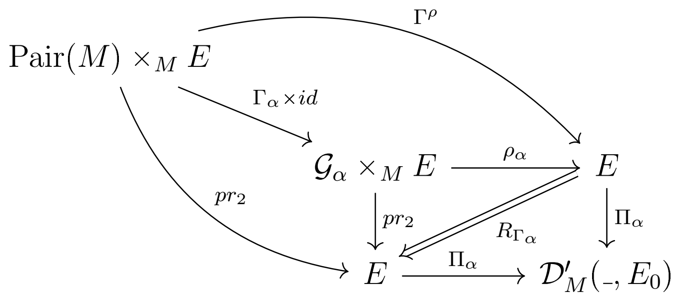
(5) $\Pi : E \longrightarrow \mathcal{D}'_M(\_, E_0)$ is a family $\Pi_\alpha : E_\alpha \longrightarrow \mathcal{D}'_M(\_, E_0), \alpha \in A$ of maps from the total space of the vector bundle E to the sheaf $\mathcal{D}'_M(\_, E_0)$ [Tr06] of E_0-valued distributions on M which is linear and continuous on the fibres. For a point $x \in M$, $\Pi_x = \{\Pi_{\alpha_x}, \alpha \in A\}$ defines a family of distributions supported in a neighborhood of x.

(6) $\Gamma : \mathrm{Pair}(M) \ast\!\!\longrightarrow \mathcal{G}$ is a family $\Gamma_\alpha : \mathrm{Pair}(M) \ast\!\!\longrightarrow \mathcal{G}_\alpha, \alpha \in A$ of direct connections Γ_α on $\mathcal{G}_\alpha$.

It induces a direct connection $\iota_\alpha \circ \Gamma_\alpha$ on $\mathrm{Iso}(E_\alpha)$, as in the commutative diagram

$$\begin{array}{ccc} \mathrm{Pair}(M) & \xrightarrow{\ \Gamma\ } & \mathcal{G}_\alpha \\ & \searrow^{\iota \circ \Gamma} & \big\downarrow \iota \\ & & \mathrm{Iso}(E_\alpha) \end{array} , \tag{130}$$

with local maps $\Gamma^\rho_\alpha \equiv \iota_\alpha \circ \Gamma_\alpha : \mathrm{Pair}(M) \times E_\alpha \twoheadrightarrow E_\alpha, \alpha \in A$. The obstruction to Γ^ρ_α defining a groupoid action of $\mathrm{Pair}(M)$ on E_α is encoded in the following non-commutative diagram



(131)

(7) We require continuity of the map $x \longmapsto \Pi_x$ for the convergence of distributions.

Remark 5.2. For the sake of simplicity, as in (132) we shall often drop the subscript α in Π and Γ^ρ.

Combining the continuity assumption (7) on the map $x \mapsto \Pi_x$ with $\lim_{y_0 \to x_0} \Gamma^\rho(y_0, x_0) f(x_0) = f(x_0)$ yields

$$\lim_{y_0 \to x_0} \Pi_{y_0} \circ \Gamma^\rho(y_0, x_0) f = \Pi_{x_0} f \quad \forall f \in E^\alpha, \tag{132}$$

which we view as a Γ- *invariance* of Π in the limit.

Yet, one cannot expect the exact "Γ- invariance" of Π expressed by

$$\Pi_{x_0} = \Pi_{y_0} \circ \Gamma^\rho(y_0, x_0), \tag{133}$$

– assumed to hold for regularity structures on $\mathbb{R}^n$, cf. [H14, Definition 2.1] or [H2, Definition 2.1]– to hold for regularity structures on general vector bundles.

For regularity structure on a manifold studied in [DDD19], a **transport precision** of the model (Π, Γ) is defined, which as well as the uniform rescaling properties of Π_x and the transport regularity of $\Gamma(y, x)$, takes into account the discrepancy between Π_x and $\Pi_y \Gamma(y, x)$.

The following proposition expresses the obstruction to the "Γ-invariance of Π"

$$\Delta_{y_0, x_0}(\Pi, \Gamma^\rho) := \Pi_{y_0} \circ \Gamma^\rho(y_0, x_0) - \Pi_{x_0}, \quad \forall (x_0, y_0) \in \mathcal{U}_\Delta(x_0) \tag{134}$$

in terms of a curvature term for the direct connection Γ^ρ.

Proposition 5.3. *With the notations of the proposition, we have*

$$\begin{aligned} & \Delta_{z_0, x_0}(\Pi, \Gamma^\rho) - \Delta_{y_0, x_0}(\Pi, \Gamma^\rho) + \Delta_{z_0, y_0}(\Pi, \Gamma^\rho)\, \Gamma^\rho(y_0, x_0) \\ = \;& \Pi_{z_0} \circ \left(\widetilde{R}^{\Gamma^\rho}(y_0, x_0, z_0) - \mathrm{Id}_{z_0} \right) \Gamma^\rho(z_0, x_0), \end{aligned} \tag{135}$$

where

$$\widetilde{R}^{\Gamma^\rho}(z, y, x) := \Gamma^\rho(x, z)\, \Gamma^\rho(z, y) \Gamma^\rho(x, y)^{-1} \in \mathrm{End}(E_x).$$

If Γ is natural in the sense of Definition 2.19, namely if $\Gamma^{-1}(y, x) := \Gamma(x, y)^{-1} = \Gamma(y, x)$, then

$$\Delta_{z_0, x_0}(\Pi, \Gamma^\rho) - \Delta_{y_0, x_0}(\Pi, \Gamma^\rho) + \Delta_{z_0, y_0}(\Pi, \Gamma^\rho)\, \Gamma^\rho(y_0, x_0) = \Pi_{z_0} \circ \left(R^{\Gamma^\rho}(y_0, x_0, z_0) - \mathrm{Id}_{z_0} \right) \Gamma^\rho(z_0, x_0), \tag{136}$$

where $R^{\Gamma^\rho}(z, y, x) = \Gamma(z, x)^{-1}\, \Gamma(z, y) \Gamma(y, x)$ is the curvature of Γ^ρ defined in eq. (35). Consequently, the "Γ-invariance" of Π holds if Γ is flat. Conversely, if Π is injective, "Γ-invariance" of Π implies flatness of Γ.

Proof. Let us compute the obstruction to the "Γ-invariance" of Π:

$$\begin{aligned}
\Delta_{z_0,y_0}(\Pi,\Gamma^\rho) &= \Pi_{z_0}\circ\Gamma^\rho(z_0,y_0)-\Pi_{y_0}\\
&= \left(\Pi_{z_0}\circ\Gamma^\rho(z_0,y_0)\,\Gamma^\rho(y_0,x_0)-\Pi_{y_0}\circ\Gamma^\rho(y_0,x_0)\right)\,\Gamma^\rho(y_0,x_0)^{-1}\\
&= \left(\Pi_{z_0}\circ\Gamma^\rho(z_0,y_0)\,\Gamma^\rho(y_0,x_0)-\Pi_{x_0}-\Delta_{y_0,x_0}(\Pi,\Gamma^\rho)\right)\,\Gamma^\rho(y_0,x_0)^{-1}\\
&= \left(\Pi_{z_0}\circ\Gamma^\rho(z_0,y_0)\,\Gamma^\rho(y_0,x_0)-\Pi_{z_0}\,\Gamma^\rho(z_0,x_0)+\Delta_{z_0,x_0}(\Pi,\Gamma^\rho)-\Delta_{y_0,x_0}(\Pi,\Gamma^\rho)\right)\,\Gamma^\rho(y_0,x_0)^{-1}\\
&= \left(\Pi_{z_0}\circ\Gamma^\rho(z_0,y_0)\,\Gamma^\rho(y_0,x_0)\,\Gamma^\rho(z_0,x_0)^{-1}-\Pi_{z_0}\right)\,\Gamma^\rho(z_0,x_0)\,\Gamma^\rho(y_0,x_0)^{-1}\\
&+ \left(\Delta_{z_0,x_0}(\Pi,\Gamma^\rho)-\Delta_{y_0,x_0}(\Pi,\Gamma^\rho)\right)\,\Gamma^\rho(y_0,x_0)^{-1}.
\end{aligned}$$

Hence,

$$\Delta_{z_0,x_0}(\Pi,\Gamma^\rho)-\Delta_{y_0,x_0}(\Pi,\Gamma^\rho)+\Delta_{z_0,y_0}(\Pi,\Gamma^\rho)\,\Gamma^\rho(y_0,x_0)=\Pi_{z_0}\circ\left(\widetilde{R}^{\Gamma^\rho}(y_0,x_0,z_0)-\mathrm{Id}_{z_0}\right)\,\Gamma^\rho(z_0,x_0).$$

If $R^{\Gamma^\rho}=\mathrm{Id}$, an easy computation yields $\widetilde{R}^{\Gamma^\rho}=\mathrm{Id}$, which inserted in eq. (135) gives rise to $\Delta_{z_0,x_0}(\Pi,\Gamma^\rho)-\Delta_{y_0,x_0}(\Pi,\Gamma^\rho)+\Delta_{z_0,y_0}(\Pi,\Gamma^\rho)\,\Gamma^\rho(y_0,x_0)=0$ for any $(y_0,z_0)\in\mathcal{U}_\Delta(x_0)^2$. Taking $y_0=x_0$ then leads to $\Delta_{z_0,x_0}(\Pi,\Gamma^\rho)=0$.

Conversely, assuming "Γ-invariance" of Π, we have

$$\Pi_{z_0}\circ\left(\widetilde{R}^{\Gamma^\rho}(y_0,x_0,z_0)-\mathrm{Id}_{z_0}\right)\,\Gamma^\rho(z_0,x_0)=0$$

which implies that $\Pi_{z_0}\circ\left(\widetilde{R}^{\Gamma^\rho}(y_0,x_0,z_0)-\mathrm{Id}_{z_0}\right)=0$ so that if Π is injective, then $\widetilde{R}^{\Gamma^\rho}(y_0,x_0,z_0)=\mathrm{Id}_{z_0}$ which in turn implies $\Gamma^\rho=\mathrm{Id}$. □

5.2. **Geometric polynomial pre-regularity structure.** Underlying a geometric pre-regularity structure $(A,E,\mathcal{G},\rho)$ on a manifold M as in Definition 5.1, there is an abstract regularity structure (A,T,G) which generalises Hairer's abstract set up [H14] and relates to the polynomial regularity structures built in [DDD19]:

- The model space T is the model fibre of the vector bundle $E\to M$,
- The structure group G is the vertex group of the gauge groupoid $\mathcal{G}\rightrightarrows M$.

We now discuss polynomial regularity structures, which correspond to the case of a jet bundle $E=J^nE_0\to M$.

Theorem 5.4. *Let M be a d-dimensional smooth manifold. We consider the initial data (E_0,P_0,G_0,ρ_0) where*

- *$E_0\to M$ is a real vector bundle with typical fibre $V\cong\mathbb{R}^r$,*
- *G_0 is a Lie group endowed with a faithful representation ρ_0 on V, inducing an inclusion $\iota:G_0\hookrightarrow GL(V)$ of Lie groups,*
- *$P_0\to M$ is a principal G_0-bundle such that $E_0=P_0\times_{G_0}V$.*

(1) The data $(A,E,\mathcal{G},\rho)$ given by
  - *the (finite) index set $A=[[0,n]]$ where n is a given non negative integer,*
  - *the (finite limit) jet bundle $E:=\varprojlim_{k\in A}J^kE_0\cong J^nE_0\to M$ of E_0,*
  - *the (finite limit) jet prolongation $\mathcal{G}:=\varprojlim_{k\in A}J^k\mathcal{G}(P_0)=J^n\mathcal{G}(P_0)\cong\mathcal{G}(W^nP_0)$ of the gauge groupoid of the principal G_0-bundle P_0,*
  - *the linear representation ρ of $\mathcal{G}$ on E given by the jet prolongation of ρ_0, equivalent to an inclusion $\mathcal{G}\hookrightarrow\mathrm{Iso}(E)$,*

yields a geometric pre-regularity structure on M in the sense of Definition 5.1, which we call **geometric polynomial pre-regularity structure**. *Its underlying abstract regularity structure is (A, T, G) where*

- $T = T^n_d V = J^n_0(\mathbb{R}^d, V) \cong \mathbb{R}_n[X_1, ..., X_d] \otimes V$ *is the typical fibre of $J^n E_0$, cf. (54),*
- $G = W^n_d G_0 = GL^n_d(\mathbb{R}) \ltimes T^n_d G_0$ *is the vertex group of the groupoid $\mathcal{G}$, cf. (56).*

Proof.

(1) We first observe that the initial data includes the following structure:

- the frame bundle $FM \to M$, which is a principal $GL_d(\mathbb{R})$-bundle, together with its associated frame groupoid $\mathrm{Iso}(TM) = \mathcal{G}(FM) \cong J^1\,\mathrm{Pair}(M)$ (cf. §1.6),
- the frame bundle $FE_0 \to M$ of $E_0 \to M$, which is a principal $GL(V) \cong GL_r(\mathbb{R})$-bundle, together with its associated frame groupoid $\mathcal{G}(FE_0) = \mathrm{Iso}(E_0)$, which comes with a natural faithful and linear left groupoid action (cf. §1.4) given by the evaluation map $\mathrm{ev}_0 : \mathrm{Iso}(E_0) \times_M E_0 \to E_0$,
- the principal bundle $P_0 \to M$, which is a reduction of the frame bundle FE_0 with structure group $G_0 \subset GL(V)$, together with its gauge groupoid $\mathcal{G}(P_0)$ which is a reduction of the frame groupoid $\mathrm{Iso}(E_0)$. The inclusion of groupoids $\iota_0 : \mathcal{G}(P_0) \hookrightarrow \mathrm{Iso}(E_0)$ composed with ev_0 determines a faithful left linear groupoid action $\rho_0 : \mathcal{G}(P_0) \times_M E_0 \to E_0$:

$$\begin{array}{ccc} \mathcal{G}(P_0) \times_M E_0 & \xhookrightarrow{\iota_0 \times \mathrm{id}} & \mathrm{Iso}(E_0) \times_M E_0 \\ & \searrow^{\rho_0} & \downarrow \mathrm{ev}_0 \\ & & E_0 \end{array}$$

.

(2) We now fix $n \in \mathbb{N}$ and apply the n-jet prolongation to the initial data:

- With the notations of §3.2, let us consider the jet prolongation of the pair groupoid $J^n\,\mathrm{Pair}(M) \cong \mathcal{G}(F^n M)$, where $F^n M = \mathrm{inv} J^n(\mathbb{R}^d, M)$ is the n-frame bundle of M (which is a principal $GL^n_d(\mathbb{R})$-bundle with $GL^n_d(\mathbb{R}) = \mathrm{inv} J^n_0(\mathbb{R}^d, \mathbb{R}^d)_0$).
- Consider the jet bundle $J^n E_0$ with fibre $T^n_d V = J^n_0(\mathbb{R}^d, V) \cong J^n_0(\mathbb{R}^d, \mathbb{R}^r) = P^n_{d,r}$, together with its frame bundle $FJ^n E_0$, with structure group $GL(T^n_d V) \cong GL(P^n_{d,r})$. Its associated frame groupoid $\mathrm{Iso}(J^n E_0) \cong \mathcal{G}(FJ^n E_0)$ naturally acts on $J^n E_0$ by the evaluation map $\mathrm{ev} : \mathrm{Iso}(J^n E_0) \times_M J^n E_0 \to J^n E_0$.
- The frame bundle $FJ^n E_0$ is not a jet prolongation (cf. Example 3.4), but it admits a reduction to the jet bundle $W^n(FE_0) = F^n M \times_M J^n(FE_0)$, which is a principal $W^n_d GL(V)$-bundle with structure group given by the jet group $W^n_d GL(V) = GL^n_d(\mathbb{R}) \ltimes T^n_d GL(V) \subset GL(T^n_d V)$. Therefore, there is a canonical inclusion of gauge groupoids (cf. Example 3.2)

$$\kappa = (\kappa_0)^{(n)} : J^n \mathrm{Iso}(E_0) \cong \mathcal{G}(W^n FE_0) \hookrightarrow \mathrm{Iso}(J^n E_0) \cong \mathcal{G}(FJ^n E_0) \tag{137}$$

which induces an action of $J^n \mathrm{Iso}(E_0)$ on $J^n E_0$ by composition with ev, namely

$$\begin{array}{ccc} J^n \mathrm{Iso}(E_0) \times_M J^n E_0 & \xhookrightarrow{\kappa \times \mathrm{id}} & \mathrm{Iso}(J^n E_0) \times_M J^n E_0 \\ & \searrow & \downarrow \mathrm{ev} \\ & & J^n E_0 \end{array}$$

- Since E_0 has a G_0-structure, the jet bundle J^nE_0 has a $W^n_dG_0$-structure, where $W^n_dG_0 = GL^n_d(\mathbb{R}) \ltimes T^n_dG_0$ so that the principal $W^n_dGL(V)$-bundle W^nFE_0 further reduces to the principal $W^n_dG_0$-bundle $W^nP_0 = F^nM \times_M J^nP_0$ (cf. eq. (60)). The jet prolongation of the inclusion ι_0 gives an inclusion of jet groupoids (cf. eq. (80))

$$\iota = (\iota_0)^{(n)} : J^n\mathcal{G}(P_0) \cong \mathcal{G}(W^nP_0) \hookrightarrow J^n\mathrm{Iso}(E_0) \cong \mathcal{G}(W^nFE_0),$$

and therefore an associated faithful linear groupoid action of $J^n\mathcal{G}(P_0)$ on J^nE_0. This action commutes with that of $J^n\mathrm{Iso}(E_0)$ and of $\mathrm{Iso}(J^nE_0)$ on J^nE_0 and it coincides with the jet prolongation $\rho = (\rho_0)^{(n)}$ of ρ_0. In other words, we have a commutative diagram

$$\begin{array}{ccccc} J^n\mathcal{G}(P_0) \times_M J^nE_0 & \xrightarrow{\iota\times\mathrm{id}} & J^n\mathrm{Iso}(E_0) \times_M J^nE_0 & \xrightarrow{\kappa\times\mathrm{id}} & \mathrm{Iso}(J^nE_0) \times_M J^nE_0 \\ & \searrow^{\rho} & \downarrow & \swarrow_{\mathrm{ev}} & \\ & & J^nE_0 & & \end{array}$$

Setting $\mathcal{G} = J^n\mathcal{G}(P_0) \cong \mathcal{G}(W^nP_0)$, the resulting structure $(A, E, \mathcal{G}, \rho)$ satisfies the conditions of Definition 5.1, since the action ρ of the jet prolongation $J^n\mathcal{G}(P_0)$ on the jet bundle J^nE_0 is itself a jet prolongation and therefore preserves the involved projective systems.

□

For a trivial vector bundle $E_0 = M \times \mathbb{R}$ of rank 1 we get back the polynomial regularity structures of on a Riemannian manifold. Specialising to $\mathbb{R}^d$ equipped with the canonical Euclidean metric yields the polynomial regularity structures of [H14].

Example 5.5. Let M be a Riemannian manifold equipped with a connection ∇^M (e.g. $M = \mathbb{R}^d$ equipped with the canonical metric and a Riemannian connection). If for initial data (E_0, P_0, G_0, ρ_0) we choose

- the trivial vector bundle $E_0 = M \times \mathbb{R}$ of rank 1,
- the frame bundle $P_0 = FE_0$ with structure group $G_0 = GL_1(\mathbb{R})$ and $\iota_0 = \mathrm{Id}$,

then the geometric polynomial regularity structure $(A, E, \mathcal{G}, \rho)$ given in Theorem 5.2 yields back the polynomial regularity structure of [DDD19] and in particular, that of [H14] if $M = \mathbb{R}^d$

- The typical fibre of $E = J^nE_0$ is the graded vector space $T := T^n_d\,\mathbb{R} \cong \mathbb{R}_n[X_1, ..., X_d]$ of real polynomials of degree n in d variables,
- The structure group is the vertex group $W^n_dGL_1(\mathbb{R}) \cong GL^n_d(\mathbb{R}) \ltimes T^n_dGL_1(\mathbb{R})$ of the jet prolongation of the frame groupoid $J^n\mathrm{Iso}(E_0) \cong \mathcal{G}(W^nFE_0) \subsetneq \mathrm{Iso}(J^nE_0)$, which acts as the identity on the homogeneous component of T of degree 0, and as an isomorphism on the homogeneous component of T of degree n.

  The n-jet group elements in $W^n_dGL_1(\mathbb{R})$ are jet prolongations of 1-jets in $W^1_dGL_1(\mathbb{R}) \cong GL^1_d(\mathbb{R}) \ltimes T^1_dGL_1(\mathbb{R})$, where the component $GL^1_d(\mathbb{R}) = GL_d(\mathbb{R})$ encodes the Jacobian matrix of a change of local coordinates in M around any given point.

The identification with the known polynomial structures of [H14] and [DDD19] requires further remarks:

- For $M = \mathbb{R}^d$, the structure group chosen in [H14] is the jet prolongation of the subgroup $(\mathbb{R}^d, +) \cong \{1\} \times T^1_d GL_1(\mathbb{R}) \subset W^1_d GL_1(\mathbb{R})$ resulting from the choice of fixing the local coordinates at each point of $\mathbb{R}^d$ (which moreover are global).
- Note the construction of the polynomial structure on a manifold M carried out in [DDD19] uses the bundle $\oplus_{k=0}^n S^k(T^*M)$ which amounts to $\oplus_{k=0}^n S^k(T^*M) \otimes E$ in the case $E = M \times \mathbb{R}$. We saw in eq. (54) that the n-th jet prolongation J^nE of a real vector bundle $\pi : E \to M$ of rank r on a d-dimensional manifold M, is like $\oplus_{k=0}^n S^k(T^*M) \otimes E$, a vector bundle modelled on $\oplus_{k=0}^n S^k((\mathbb{R}^d)^*) \otimes \mathbb{R}^r$.

  Yet their structure groups apriori differ. If E has structure group $\mathrm{GL}_r(\mathbb{R})$, then the vector bundle J^nE has structure group $W^n_d \mathrm{GL}^n_r(\mathbb{R})$ described in eq. (59), whereas the vector bundle $\oplus_{k=0}^n S^k(T^*M) \otimes E$ has structure group $\mathrm{GL}_d(\mathbb{R}) \times \mathrm{GL}_r(\mathbb{R})$. However, one can reduce the $W^n_d \mathrm{GL}^n_r(\mathbb{R})$ structure group of J^nE to $\mathrm{GL}_d(\mathbb{R}) \times \mathrm{GL}_r(\mathbb{R})$ by means of a connection ∇^M on M and a connection ∇ on E via the maps defined in eq. (**??**), which yield isomorphims of vector bundles. These isomorphisms are compatible with the canonical projections $\pi^n_{n-1} : J^nE \to J^{n-1}E$ (cfr. (51)) and lead to a reduction of J^nE to the bundle $\oplus_{k=0}^n S^k(T^*M) \otimes E$ with structure group $\mathrm{GL}_d(\mathbb{R}) \times \mathrm{GL}_r(\mathbb{R})$.

  Note that a reduction of the structure group of J^nE by means of a reduction of E requires flatness [Fr76, Definition (5.1), Theorem (5.2) and Theorem (5.18)], combined with a reduction of the frame bundle FM which requires that the manifold carries a linear structure i.e., that it has a covering with locally constant transition functions [Fr76, Definition (7.11) and Theorem (7.12)].

Theorem 5.6. *Let M, the initial data (E_0, P_0, G_0, ρ_0) and the geometric polynomial regularity structure $(A, E, \mathcal{G}, \rho)$ be as in Theorem 5.2. Assume that*

- *the Riemannian manifold M is equipped with a connection ∇^M,*
- *the vector bundle E_0 is endowed with a linear connection ∇_0 associated to a connection 1-form $\omega_0 \in \Omega^1(P_0, \mathfrak{g}_0)$ on the principal G_0-bundle $P_0 \to M$, where $\mathfrak{g}_0$ is the Lie algebra of G_0,*
- *the groupoid $\mathcal{G}(P_0)$ is endowed with a direct connection $\Gamma_0 : \mathrm{Pair}(M) \to \mathcal{G}(P_0)$ whose infinitesimal connection is ω_0.*

Then, the data $(\Pi, \Gamma) = \{(\Pi^n, \Gamma^n), n \in \mathbb{Z}_{\geqslant 0}\}$:

- *with $\Pi^n : J^nE_0 \to C^\infty_M(\_, E_0) \subset \mathcal{D}'_M(\_, E_0)$ defined on the fibre above any point $x_0 \in M$ by the linear map $\Pi^n_{x_0} : J^n_{x_0}E_0 \to C^\infty_M(U_{x_0}, E_0) \subset \mathcal{D}'_M(U_{x_0}, E_0)$, where U_{x_0} is a given normal neighborhood of x_0, by*

$$\Pi^n_{x_0}(j^n_{x_0}f)(x) = \Gamma_0(x, x_0) \sum_{k=0}^{n} \frac{1}{k!} \frac{d^k}{dt^k} \left(\Gamma_0^{-1}(x, c_x(t))(f(c_x(t)))\right)\Big|_{t=0}, \tag{138}$$

  as in eq. (115).
- *and $\Gamma^n : \mathrm{Pair}(M) \to J^n\mathcal{G}(P_0)$ the direct connection*

$$\Gamma^n(y_0, x_0)\, j^n_{x_0}f := j^n_{y_0}\left(y \mapsto \Pi^n_{x_0}(j^n_{x_0}f)(y)\right)$$

  given by $\widetilde{\Gamma}^{(n)}$ defined in eq. (123),

yields a geometric pre-model for $(A, E, \mathcal{G}, \rho)$ on E_0 in the sense of Definition 5.1. In particular, it fits into the diagram (131).

Moreover,

(1) *(cfr. [DDD19, Lemma 90]) A continuous section f of E_0 is γ- Hölder continuous for some $\gamma > 0$ iff it is n-times continuously differentiable with $n = [\gamma]$ the integral part of γ and for any norm $\|\cdot\|$ and any direct connection Γ^n on J^nE_0, there is a constant C such that*

$$\|\Gamma^n(y,x)\, j^n_x f - j^n_y f\| \leqslant C\, d(x,y)^{\gamma - n}, \quad \forall (x,y) \in \mathcal{U}_\Delta, \tag{139}$$

where $d(x,y)$ is the geodesic distance between x and y.

(2) *The Γ^n-invariance of Π^n for any $j^n_{x_0} f$ built from an arbitrary local section of E_0 above any point $x_0 \in M$:*

$$\Pi^n_{y_0}\left(\Gamma^n(y_0,x_0)\cdot j^n_{x_0} f\right) = \Pi^n_{x_0}\left(j^n_{x_0} f\right), \quad \forall y_0 \in U_{x_0} \quad \forall n \in \mathbb{Z}_{\geqslant 0} \tag{140}$$

implies the flatness of Γ.

Proof. The fact that the family $\{(\Pi^n, \Gamma^n), n \in \mathbb{Z}_{\geqslant 0}\}$ defines a geometric pre-model follows from Proposition 4.18 which says that Γ^n indeed defines a direct connection, combined with the continuity of the map $x \mapsto \Pi_x$. The latter easily follows from

$$\begin{aligned}
\lim_{y_0 \to x_0} \Pi^n_{y_0}(j_{y_0} f)(x) &= \lim_{y_0 \to x_0} \left(\sum_{k=0}^{n} \frac{1}{k!} \nabla^k_0 f(y_0) \left(\exp^{-1}_{y_0}(x)^{\otimes k}\right) \right) \\
&= \sum_{k=0}^{n} \frac{1}{k!} \nabla^k_0 f(x_0) \left(\exp^{-1}_{x_0}(x)^{\otimes k}\right) \\
&= \Pi^n_{x_0}(j_{x_0} f)(x)
\end{aligned}$$

for any local section f of E_0 in a neighborhood of x_0.

- To prove (1) we use a local trivialisation. Let us first recall that the vector bundle J^nE_0 is modelled on $J^n_0(\mathbb{R}^d, \mathbb{R}^r) \cong \bigoplus_{k=0}^n S^k((\mathbb{R}^d)^*) \otimes \mathbb{R}^r$ with $d = \dim(M)$. We prove (139) using a local trivialisation $E_{|U_x} \to \varphi_x(U_x) \times \mathbb{R}^r \subset \mathbb{R}^d \times \mathbb{R}^r$, $(z, f_z) \mapsto (\varphi_x(z), \Phi_x f_z)$ with $\varphi_x(x) = 0 \in \mathbb{R}^d$, which induces a local trivialisation of J^nE_0

$$\begin{aligned}
J^nE_0|_{U_x} &\longrightarrow J^n(\varphi_x(U_x) \times \mathbb{R}^r) \\
(z, j^n_z f) &\longmapsto \left(\varphi_x(z), j^n_{\varphi_x(z)}(\Phi_x \circ (\varphi_x)_* f)\right)
\end{aligned}$$

  and a local description of Γ^n

$$\begin{aligned}
J^nE^* \boxtimes J^nE|_{U_x \times U_x} &\longrightarrow \mathrm{End}\left(J^n(\varphi_x(U_x) \times \mathbb{R}^r)\right)) \\
((z,w), \Gamma^n_{zw}) &\longmapsto \left(\varphi_x(z), \varphi_x(w), \underbrace{j^n_{\varphi_x(z)}(\Phi_x \circ (\varphi_x)_*)\, \Gamma^n_{zw}\, j^n_{\varphi_x(w)}\left(\varphi_x^* \circ \Phi_x^{-1}\right)}_{A^n_{\varphi_x(z)\,\varphi_x(w)}} \right),
\end{aligned}$$

  with $A^n_{\varphi_x(y)\,0} = \mathrm{Id}_0 + O(|\varphi_x(y)|) \in \mathrm{End}(J^n(\varphi_x(U_x) \times \mathbb{R}^r))$ as a consequence of the differentiability of Γ^n along the diagonal.

  In this local trivialisation, the difference $\Gamma^n(y,x)\, j^n_x f - j^n_y f$ reads

$$\begin{aligned}
& j^n_{\varphi_x(y)}(\Phi_x \circ (\varphi_x)_*)\, \Gamma^n_{yx}\; j^n_x f - j^n_{\varphi_x(y)}\left(\Phi_x \circ (\varphi_x)_*\right)\, j^n_y f \\
= \;& A^n_{\varphi_x(y)\,0}\, j^n_0(\Phi_x \circ (\varphi_x)_* f) - j^n_{\varphi_x(y)}\left(\Phi_x \circ (\varphi_x)_* f\right) \\
= \;& \left(A^n_{\varphi_x(y)\,0} - \mathrm{Id}_0\right)\, j^n_0(\Phi_x \circ (\varphi_x)_* f) + j^n_0(\Phi_x \circ (\varphi_x)_* f) - j^n_{\varphi_x(y)}\left(\Phi_x \circ (\varphi_x)_* f\right).
\end{aligned}$$

A norm $z \mapsto \|j^n_z f\|$ on the vector bundle $J^n E_0 \to M$ induces a norm $u \mapsto \|j^n_u \tilde{f}\|_{\varphi,\Phi} := \|j^n_{\varphi^{-1}(u)}(\varphi_x^* \circ \Phi_x^{-1} \tilde{f})\|$ on the vector bundle $J^n(\varphi_x(U_x) \times \mathbb{R}^r)$.

With these notations, eq. (139) amounts to the existence of a constant C such that

$$\|j^n_{\varphi_x(y)}(\Phi_x \circ (\varphi_x)_*)\, \Gamma^n_{yx}\, j^n_x f - j^n_{\varphi_x(y)}\,(\Phi_x \circ (\varphi_x)_*)\; j^n_y f\|_{\varphi,\Phi} \leqslant C\, |\varphi_x(y)|^{\gamma-n},$$

or equivalently to the existence of a constant D such that

$$\|j^n_0(\Phi_x \circ (\varphi_x)_* f) - j^n_{\varphi_x(y)}\,(\Phi_x \circ (\varphi_x)_* f)\,\|_{\varphi,\Phi} \leqslant D\, |\varphi_x(y)|^{\gamma-n}, \tag{141}$$

since there is a constant C' such that

$$\|\left(A^n_{\varphi_x(y)\,0} - \mathrm{Id}_0\right)\, j^n_0(\Phi_x \circ (\varphi_x)_* f)\| \leqslant C'\, |\varphi_x(y)| \leqslant C'\, |\varphi_x(y)|^{\gamma-n}$$

for y and x near enough using the fact that $1 > \gamma - n \geqslant 0$.

Thus, conditions (139) and (141) are equivalent. Since the latter is the γ-Hölder condition

$$\|j^n_0 \tilde{f} - j^n_z \tilde{f}\|_{\varphi,\Phi} \leqslant D\, |z|^{\gamma-n}$$

in local coordinates, the assertion follows.

- To prove (2), we observe that (140) reads

$$\Pi^n_{y_0} \circ j^n_{y_0}\left(\Pi^n_{x_0}(j^n_{x_0} f)\right) = \Pi^n_{x_0}(j^n_{x_0} f)$$

for any local section f of E_0 above U_{x_0}. The local map $g_n := \Pi^n_{x_0}(j^n_{x_0} f)$ defines a local section of E_0 over U_{x_0} and by (120) we have

$$\Gamma(y, y_0) \sum_{k=0}^{n} \frac{1}{k!}\, \nabla^k g_n\, \left(\exp^{-1}_{y_0} y\right)^{\otimes k} = g_n(y) \quad \forall (y_0, y) \in U^2_{x_0}, \forall n \in \mathbb{Z}_{\geqslant 0}\,.$$

For $n = 0$, the above equation reads $\Gamma(y, y_0)\, g_0(y_0) = g_0(y)$. Hence $\Gamma(z, y)\, \Gamma(y, y_0)\, g_0(y_0) = \Gamma(z, y)\, g_0(y) = g_0(z)$ which in turn implies that

$$\Gamma(y_0, z)\, \Gamma(z, y)\, \Gamma(y, y_0)\, g_0(y_0) = \Gamma(y_0, z)\, g_0(z) = g_0(y_0).$$

Since this holds for any $g_0(y_0) \in E_{y_0}$ (which actually coincides with $f(x_0)$), it follows that $\Gamma(y_0, z)\, \Gamma(z, y)\, \Gamma(y, y_0) = 1_{y_0}$, which implies the flatness of Γ.

□

6. Appendix: Prounipotent gauge groupoids with direct connections

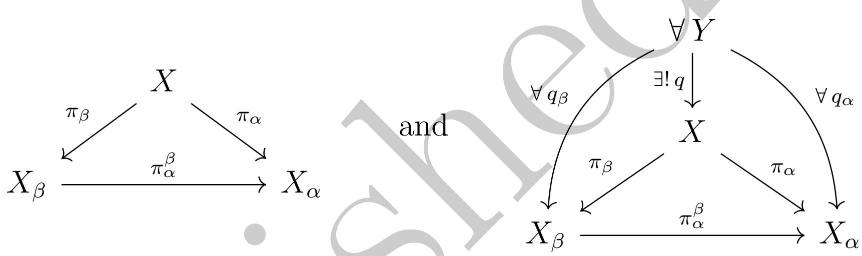
In order to keep track of the grading consistently with the analytic and the geometric setup, we need the notion of a *projective limit groupoid.*

Projective limits arise in infinite dimensional geometry, typically as inverse limits of Banach or Hilbert manifolds, Lie groups or vector bundles. Also, jet spaces, jet groups and jet bundles naturally fit into projective systems, which is what motivates this appendix.

6.1. **Projective (inverse) limits.** Let $\mathcal{C}$ be a category and let $(A, \leqslant)$ be a partially ordered set of indices. An A-**projective system** in $\mathcal{C}$ is a collection $(X_\alpha)_{\alpha\in A}$ of objects in $\mathcal{C}$ together with a collection of **connecting** or **bonding** maps $\pi_\alpha^\beta : X_\beta \to X_\alpha$ for any indices $\alpha \leqslant \beta$, such that [McL97, Chapter III, §4]

$$\pi_\alpha^\alpha = \mathrm{Id} : X_\alpha \to X_\alpha \qquad \text{and} \qquad \pi_\alpha^\beta \circ \pi_\beta^\gamma = \pi_\alpha^\gamma \qquad \forall \alpha \leqslant \beta \leqslant \gamma. \tag{142}$$

The **projective limit** of the family $(X_\alpha)_{\alpha\in A}$ is an object X in $\mathcal{C}$, necessarily unique and usually denoted by $\varprojlim_\alpha X_\alpha$ or $\varprojlim_{\alpha\in A} X_\alpha$, together with a collection of connecting maps $\pi_\alpha : X \to X_\alpha$ which make the following diagrams commute (where the second one is a universal property):



. (143)

The projective limit exists in the category of sets [McL97, Chapter V, §1] and can be realised as the subset of the cartesian product $\prod_\alpha X_\alpha$ made of tuples compatible with the connecting maps π_α^β, namely

$$\varprojlim_\alpha X_\alpha = \Big\{ (a_\alpha) \in \prod_\alpha X_\alpha \mid a_\alpha = \pi_\alpha^\beta(a_\beta)\ \forall \alpha \leqslant \beta \Big\}. \tag{144}$$

Using the forgetful functor to sets, the projective limit is then constructed in several basic categories (topological spaces, groups, algebras, modules over a fixed ring, etc), in requiring that the bonding maps π_α^β be surective morphisms in the category and showing that on the projective limit of the underlying sets one can define the desired extra structure (topology, group law, etc).

Example 6.1. The vector space of formal series $\mathbb{R}[[x]]$ is the projective limit of the projective system $(X_n)_{n\in\mathbf{N}}$ where $X_n = \mathbb{R}[x]/(x^n)$ is the quotient of the polynomial algebra by the ideal generated by x^n, with connecting maps $\pi_m^n : \mathbb{R}[x]/(x^n) \to \mathbb{R}[x]/(x^m)$ induced by the identity on $\mathbb{R}[x]$ for any $m \leqslant n$.

Example 6.2. The vector space of formal series $\mathbb{R}[[x_1, \cdots, x_d]]$ in d variables is the projective limit of the projective system $(X_n)_{n\in\mathbf{N}}$ where $X_n = \mathbb{R}[x_1, \cdots, x_d]/(\mathfrak{m}^n)$, with $\mathfrak{m} := \mathrm{Ker}(\epsilon)$ corresponding to the augmentation ideal, of dimension 1, given by the kernel of the counit $\epsilon : \mathbb{R}[x_1, \cdots, x_d] \to \mathbb{R}$ defined by zero everywhere except on $\mathbb{R}.1$ where it is the identity map.

A **prounipotent group** is a projective limit $G = \varprojlim_\alpha G_\alpha \rightrightarrows M$ of a projective system $(G_\alpha, \pi_\alpha^\beta)_{\alpha \leqslant \beta}$ of groups satisfying eqs. (142), (143) and eq. (144) functorially lifted to groups.

Example 6.3. The n**-jet groups** (see [KMS, §12.6]) defined in eq. (45) define a projective system :

$$GL_d^n(\mathbb{R}) \xrightarrow{\pi_{n-1}^n} G_d^{n-1}(\mathbb{R}) \xrightarrow{\pi_{n-2}^{n-1}} \cdots \longrightarrow GL_d^1(\mathbb{R}) \to 1,$$

6.2. **Projective (inverse) limits of vector bundles and of principal bundles.** We call **projective limit of vector bundles** on M a vector bundle E obtained as projective limit

$$E = \varprojlim_\alpha E_\alpha \to M \tag{145}$$

of a projective system $(E_\alpha, \pi_\alpha^\beta)_{\alpha \leqslant \beta}$ of vector bundles on M satisfying eqs. (142), (143) and eq. (144) functorially lifted to vector bundles. Since the category of vector bundles over a manifold M is equivalent to that of projective modules over the ring of smooth functions $C^\infty(M)$, projective limits of vector bundles exist.

We illustrate this with projective limits of jet bundles, which are of particular interest for this paper.

Example 6.4. (see e.g. [FF03]) The collection $(J^n E^0, \pi_{n-1}^n)_{n \in \mathbb{Z}_{\geqslant 0}}$ of jets $J^n E^0$ of a vector bundle $E^0 \to M$ together with the connecting maps $\pi_{n-1}^n : J^n E^0 \to J^{n-1} E^0$ corresponding to the canonical projections of eq. (51), form a projective system. The resulting projective limit

$$J^\infty E^0 := \varprojlim_n J^n E^0 \tag{146}$$

defines a vector bundle over M.●

The projective limit

$$P = \varprojlim_\alpha P_\alpha \to M \tag{147}$$

of a projective system of principal bundles $\{P_\alpha, \alpha \in A\}$ on M of principal G_α-bundles on M satisfying eqs. (142), (143) and eq. (144) is a principal $G := \varprojlim_\alpha G_\alpha$-bundle $\pi : P \to M$.

Example 6.5. Jet prolongations of a principal G^0-bundle $P^0 \to M$ over a d-dimensional manifold M, form a projective family $\{W^n P^0, n \in \mathbb{Z}_{\geqslant 0}\}$ for the canonical projections $\pi_{n-1}^n : W^n P^0 \to W^{n-1} P^0$ and yield a prounipotent principal bundle

$$W^\infty P^0 := \varprojlim_n W^n P^0. \tag{148}$$

The collection $\{P_\alpha := F E_\alpha, \alpha \in A\}$ of frame bundles of a projective system $\{E_\alpha, \alpha \in A\}$ of vector bundles is projective. Applying eq. (147) yields

$$F \varprojlim_\alpha E_\alpha = \varprojlim_\alpha F E_\alpha.$$

Example 6.6. With the notations of eq. (146), we can apply the above equation to the projective system $\{E_n := J^n E^0, n \in \mathbb{Z}_{\geqslant 0}\}$ for some vector bundle $E^0 \to M$, which yields

$$F(J^\infty E^0) = \varprojlim_n F(J^n E^0). \tag{149}$$

Remark 6.7. Eq. (67) gives rise to the following inclusions

$$F(J^n E^0) \subsetneq W^n(FE^0), n \in \mathbb{Z}_{\geqslant 0} \quad \text{and} \quad F(J^\infty E^0) \subsetneq W^\infty(FE^0). \tag{150}$$

6.3. **Prounipotent groups and groupoids.** The projective limit of a collection $\{\mathcal{G}_\alpha \rightrightarrows M, \alpha \in A\}$ of groupoids together with a collection $\{\pi_\alpha^\beta\}$ of connecting maps which is projective i.e.,

(1) $\pi_\alpha^\beta : \mathcal{G}_\beta \to \mathcal{G}_\alpha$ is a surjective morphism of groupoids
(2) π_α^α is the identity on $\mathcal{G}_\alpha$,
(3) $\pi_\alpha^\beta \circ \pi_\beta^\gamma = \pi_\alpha^\gamma$ for all $\alpha \geqslant \beta \geqslant \gamma$.

defines a Lie groupoid $\varprojlim_\alpha \mathcal{G}_\alpha \rightrightarrows M$, which we call a **prounipotent groupoid** following the terminology used in group theory.

Example 6.8. A projective system $\{P_\alpha, \alpha \in A\}$ of principal G_α-bundles $P_\alpha \to M$, gives rise to a projective system $\{\mathcal{G}(P_\alpha), \alpha \in A\}$ of gauge groupoids and a unipotent gauge groupoid $\varprojlim_\alpha \mathcal{G}(P_\alpha)$ which is the gauge groupoid of the projective limit $\varprojlim_\alpha G_\alpha$- principal bundle $\varprojlim_\alpha P_\alpha$:

$$\mathcal{G}(\varprojlim_\alpha P_\alpha) = \varprojlim_\alpha \mathcal{G}(P_\alpha). \tag{151}$$

Its vertex group is the prounipotent group $\varprojlim_\alpha G_\alpha$ which corresponds to the projective limit of the vertex groups $\{G_\alpha, \alpha \in A\}$.

Typical examples of interest in this paper are jet groupoids $(J^n\mathcal{G}_n, \pi_{n-1}^n)_{n \in \mathbb{Z}_{\geqslant 0}}$ which form a projective family of gauge groupoids on M and we can define the prounipotent jet prolonged groupoid:

$$J^\infty \mathcal{G} := \varprojlim_n J^n \mathcal{G} \rightrightarrows M.$$

Combining eq. (154) and eq. (148) leads to the following example of relevance in this work.

Example 6.9. Given a principal G^0-bundle $P^0 \to M$, we have the following identity of prounipotent groupoids:

$$J^\infty \mathcal{G}(P^0) := \varprojlim_n J^n \mathcal{G}(P^0) \simeq \mathcal{G}(\varprojlim_n W^n P^0). \tag{152}$$

When applied to the frame bundle $P^0 = FE^0$ of a vector bundle $E^0 \to M$, this yields

$$J^n \mathrm{Iso}(E^0) \simeq \mathcal{G}(W^n FE^0) \quad \forall n \in \mathbb{Z}_{\geqslant 0} \quad \text{and} \quad \varprojlim_n J^n \mathrm{Iso}(E^0) \simeq \mathcal{G}(\varprojlim_n W^n FE^0), \tag{153}$$

since $\mathrm{Iso}(E^0) = \mathcal{G}(FE^0)$.

We now apply eq. (154) to the corresponding projective system $\{P_\alpha := FE_\alpha, \alpha \in A\}$ of frame bundles built from a projective family $\{E_\alpha, \alpha \in A\}$ of vector bundles over M. Using again the fact that frame groupoids can be viewed as gauge groupoids of frame bundles i.e. $\mathrm{Iso}(E_\alpha) = \mathcal{G}(FE_\alpha)$, this yields a projective system of frame groupoids $\{\mathrm{Iso}(E_\alpha), \alpha \in A\}$ and we have

$$\mathrm{Iso}(\varprojlim_\alpha E_\alpha) = \mathcal{G}\left(\varprojlim_\alpha FE_\alpha\right) = \varprojlim_\alpha \mathcal{G}(FE_\alpha) = \varprojlim_\alpha \mathrm{Iso}(E_\alpha). \tag{154}$$

Example 6.10. Applying this to the projective system $\{E_n := J^n E^0, n \in \mathbb{Z}_{\geqslant 0}\}$ yields

$$\mathrm{Iso}(J^\infty E^0) = \mathcal{G}\left(\varprojlim_n FJ^nE^0\right) = \varprojlim_n \mathcal{G}\left(FJ^nE^0\right) = \varprojlim_n \mathrm{Iso}(J^nE^0).$$

Remark 6.11. We recall from eq. (150) that

$$\mathrm{Iso}(J^nE^0) \subsetneq J^n\mathrm{Iso}(E^0)\,\forall n \in \mathbb{Z}_{\geqslant 0} \quad \text{and} \quad \mathrm{Iso}(J^\infty E^0) \subsetneq J^\infty \mathrm{Iso}(E^0). \tag{155}$$

Indeed, $\mathrm{Iso}(J^nE) = \mathcal{G}(F(J^nE))$, which is gauge groupoid of the frame bundle $F(J^nE)$ with structure group $GL(P^n_{d,r})$, is *not* the jet prolongation of a groupoid, because the group $GL(P^n_{d,r})$ is not the jet prolongation of a structure group and the frame bundle $F(J^nE)$ is not the jet prolongation of a principal bundle, cf. eq. (81) and eq. (77) in Example 3.4.

Recall from §1.6, that the Lie algebroid $\mathcal{LG} \to M$ of a Lie groupoid $\mathcal{G} \rightrightarrows M$ is given by the normal bundle $T\mathcal{G}_{|u(M)}/Tu(M)$ of M in $\mathcal{G}$. From the functoriality of this construction, it follows that the collection $\{\mathcal{LG}_\alpha \to M, \alpha \in A\}$ of Lie algebroids of a projective system $\{\mathcal{G}_\alpha \rightrightarrows M, \alpha \in A\}$ of Lie groupoids together with the tangent maps $T_{|u(M)}\pi^\beta_\alpha$ to the connecting maps π^β_α, is also projective and we have

$$\mathcal{L}\left(\varprojlim_\alpha \mathcal{G}_\alpha\right) = \varprojlim_\alpha \mathcal{L}\left(\mathcal{G}_\alpha\right). \tag{156}$$

6.4. **Direct connections on prounipotent groupoids.** Projective system of connections on a projective system of vector bundles were studied in [Ga88]. These are characterised by projective systems of Christoffel symbols. Here we consider projetive limits of direct connections.

We call a collection $\{\Gamma_\alpha : \mathrm{Pair}(M) \ast\!\!\longrightarrow \mathcal{G}_\alpha, \alpha \in A\}$ of direct connections on a projective system $\{\mathcal{G}_\alpha, \alpha \in A\}$ of groupoids with connecting maps π^β_α, **a projective system of connections** if it is compatible with the connecting maps in the following sense:

$$\Gamma_\alpha \pi^\beta_\alpha = \pi^\beta_\alpha \Gamma_\beta, \quad \forall (\alpha, \beta) \in A^2. \tag{157}$$

CHECK: Such a projective system yields a direct connection

$$\varprojlim_\alpha \Gamma_\alpha : \mathrm{Pair}(M) \ast\!\!\longrightarrow \varprojlim_\alpha \mathcal{G}_\alpha$$

on the prounipotent groupoid $\varprojlim_\alpha \mathcal{G}_\alpha$.

Recall from eq. (27) that the infinitesimal connection induced by a direct connection $\Gamma : \mathrm{Pair}(M) \ast\!\!\longrightarrow \mathcal{G}$ reads $\delta^\Gamma = D\Gamma_{|M} : TM \to \mathcal{LG}$. From the functoriality of this construction, it follows that the collection $\{\delta^{\Gamma_\alpha} : TM \longrightarrow \mathcal{LG}_\alpha, \alpha \in A\}$ of infinitesimal connections induced by a projective system $\{\Gamma_\alpha : \mathrm{Pair}(M) \ast\!\!\longrightarrow \mathcal{G}_\alpha, \alpha \in A\}$ of direct connections, defines a projective system whose projective limit $\varprojlim_\alpha \delta^{\Gamma_\alpha} : TM \to \varprojlim_\alpha \mathcal{LG}_\alpha$ is the infinitesimal connection of $\varprojlim_\alpha \Gamma_\alpha$ so that

$$\varprojlim_\alpha \delta^{\Gamma_\alpha} = \delta^{\varprojlim_\alpha \Gamma_\alpha}. \tag{158}$$

Natural examples are direct connections on ∞-jet groupoids, which we saw are inverse limits of groupoids.

Example 6.12. Exponential direct connections $\{\Gamma^{(n)}, n \in \mathbb{Z}_{\geqslant 0}\}$ (see eq. (103) in Definition 4.7) on the jet prolongations $\{J^n\mathcal{G}^0 \rightrightarrows M, n \in \mathbb{Z}_{\geqslant 0}\}$ of a Lie groupoid $\mathcal{G}^0 \rightrightarrows M$ with unit space a Riemannian manifold M equipped with a Riemannian connection, form a projective collection since by construction we have

$$\pi^n_{n-1}\,\Gamma^{(n)} = \Gamma^{(n-1)}\,\pi^n_{n-1} \quad \forall n \in \mathbb{Z}_{\geqslant 0}\,. \tag{159}$$

This gives rise to a direct connection

$$\Gamma^{(\infty)} := \varprojlim_n \Gamma^{(n)} : \mathrm{Pair}(M) \twoheadrightarrow \varprojlim_n J^n\mathcal{G}^0$$

on the prounipotent groupoid $\varprojlim_n J^n\mathcal{G}^0$.

Subsequently, the corresponding infinitesimal connections $\{\delta^{\Gamma^{(n)}}, n \in \mathbb{Z}_{\geqslant 0}\}$ (see Proposition 4.10) form a projective system and we have

$$\varprojlim_n \delta^{\Gamma^{(n)}} = \delta^{\Gamma^{(\infty)}}.$$

Applying this to the projective system $\{J^n\mathrm{Iso}(E^0), n \in \mathbb{Z}_{\geqslant 0}\}$ of jet prolongations of frame groupoids yields the projective limit exponential direct connection:

$$\Gamma^{(\infty)} : \mathrm{Pair}(M) \twoheadrightarrow J^\infty\mathrm{Iso}(E^0).$$

The limit direct connection $\Gamma^{(\infty)}$ induces a direct connection on the smaller groupoid $\mathrm{Iso}(J^\infty E^0)$ (see eq. (155)).

We end this appendix by noting that direct connections on a family $\{\mathrm{Iso}(J^n E^0), n \in \mathbb{Z}_{\geqslant 0}\}$ of frame groupoids of jet bundles do not necessarily form projective systems of connections.

Counterexample 6.13. Given a vector bundle $E^0 \to M$, the collection $\{\widetilde{\Gamma}^{(n)}, n \in \mathbf{N}\}$ of direct connections built on $\mathrm{Iso}(J^n E^0)$ in eq. (123) do not obey condition (159) with Γ replaced by $\widetilde{\Gamma}$.

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