Lecture 1. – Christian Bär 23/06/2018.

(M,g), (N,h) Riem mflds.

"Nice." maps $f : M \to N$

Look at functional $\frac{1}{2}\int_M |df|^2 \, d\mu_g$ and try to make that small.

Intuition: f wiggling around a lot $|df|^2$ large.

$f \equiv$ const is always an option, but might want to.

Consider the "laziest" such map in a homotopy class.

$$|df|^2 = \mathrm{tr}_g (f^* h) \; =: \; g^{ij}(h_{\alpha\beta} \circ f)\, \partial_i f^\alpha \partial_j f^\beta .$$

f (weakly) harmonic $\Leftrightarrow$ $E(f) = \frac{1}{2}\int_M |df|^2 d\mu_g$. critical pt.

Euler-Lagrange. $f \in C^2$ harmonic $\Leftrightarrow$ $\mathrm{tr}_g \nabla df = \tau(f) = 0$.

Examples:
- constant maps.
- $M = N$, $f = \mathrm{id}$.
- $M = S^1$ harmonic $\Leftrightarrow$ f geodesic.
- $N = \mathbb{R}$, harmonic $\Leftrightarrow$ $\Delta f = 0$.
- $M \hookrightarrow N$ (isometric immersion). harmonic iff M minimal.

• M, N Kähler then holomorphic $\Rightarrow$ harmonic.

2nd Variation of $E(f)$:
Ricci curvature of target mfld, then stability.
$\rightsquigarrow$ Thm by Eells–Sampson:
$\mathrm{Sec}_N \leq 0 \Rightarrow$ each homotopy class contains harmonic rep.

Good regularity!

Geometric applications:
$\mathrm{Sec}_N < 0$, $\mathrm{Ric}_M \geq 0 \Rightarrow$. f const or contained in closed geodesic

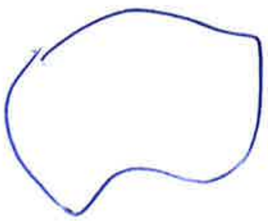
Thm (Preissman)
(N, h) with $\mathrm{Sec}_N < 0 \Rightarrow$ every nontrivial abelian subgroup of $\pi_1(N) \cong \mathbb{Z}$.

<u>Relation to harmonic maps?</u> α, β closed curves rep in $\pi_1(N)$,, commuting in $\pi_1(N)$.
$\Rightarrow$ homotopy b/w $\alpha * \beta$ and $\beta * \alpha$ yields a map from $\mathbb{T}^2 \to N$.
$\Rightarrow$ deformed into harmonic map, image in closed geo.
$\Rightarrow$ α, β diff powers of γ.

Drop $\mathrm{Sec}_N \leq 0 \Rightarrow$. no harmonic maps in general.

But $\dim M = 2$, $\pi_2(N) = 0$. $\Rightarrow$ $f: M \to N$ homotopic to harmonic map.
~~Methods here~~ are completely different: require low dim. and 2nd homotopy assumption.

Lecture 2 24/06/2019.



(M,g). $g = g_{ij}\, dx^i \otimes dx^j$

$\flat : T_pM \to T_p^*M$, $\# : T_p^*M \to T_pM$. musical iso's.

$X_p^\flat(Y_p) = g(X_p, Y_p)$, $\omega_p^\#(Y_p) =. g(\omega_p^\#, Y_p)$.

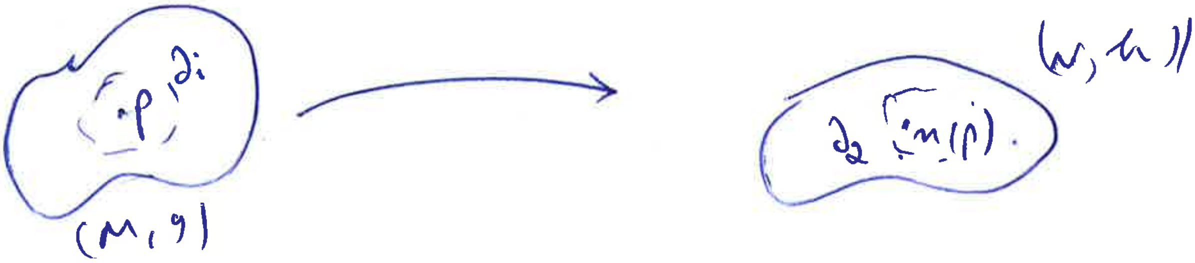
$X^\flat = g_{ij} X^j \, dx^i$. where $X = X^i \partial_i$,

$\omega^\# = g^{ij} \omega_j \partial_i$. where $\omega = \omega_j \, dx^j$.

The metric on. T^*M: $g^*(\omega, \theta) = g(\omega^\#, \theta^\#)$.

Differential $du_p : T_pM \to T_{u(p)}N$. as the follows.

$du_p(\partial_i) = \partial_i u^\alpha \partial_\alpha$. $\partial_\alpha = \partial_{y^\alpha}|_{u(p)}$.



So, $du_p = \partial_i u^\alpha \, dx^i \otimes \partial_\alpha|_{u(p)}$

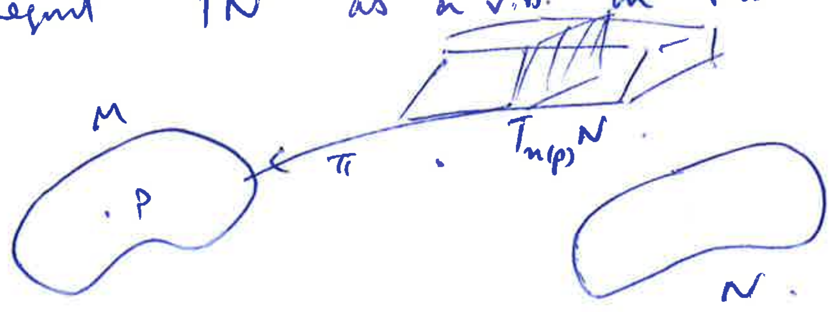
Define $\langle \cdot , \cdot \rangle$ on $T_p^*M \otimes T_{u(p)}N$:

$$\langle dx^i \otimes \partial_\alpha, dx^j \otimes \partial_\beta \rangle = \langle dx^i, dx^j \rangle \cdot \langle \partial_\alpha, \partial_\beta \rangle.$$
$$=: g^{ij}(h_{\alpha\beta} \circ u).$$

Give us a norm for du_p:

$$|du_p|^2 = \langle du_p, du_p \rangle = \partial_i u^\alpha \partial_j u^\beta g^{ij} (h_{\alpha\beta} \circ u).$$

Regard TN as a v.b. on M:



$u^{-1}TN$ v.b. on M.

<u>My note:</u> For $p \in M$, fibre over p: $T_{u(p)}N$.

Then, $u^{-1}TN = \bigsqcup_{p\in M} \{p\}\times T_{u(p)}N$.

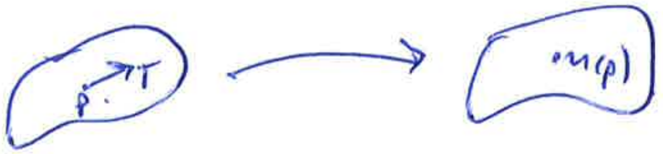
Define $e(u)(p) = \frac{1}{2}|du|^2(p) := \frac{1}{2}|du_p|^2$;

Energy: $E(u) = \int_M e(u)\, d\mu_g$.

<u>Connection induced by g:</u> ∇ L.C. for g.

Then, $\nabla^*_X \omega := (\nabla_X \omega^\#)^\flat$ connection on T^*M.

${}'\nabla$ on $u^{-1}TN$;



$${}'\nabla_{\partial_i} \partial_\alpha(u(p)) := \left(\nabla_{du_p(\partial_i)} \frac{\partial}{\partial y^\alpha}\right)(u(p)).$$

$$= \partial_i u^\alpha(p)\, \Gamma^\beta_{\alpha\gamma}(u(p))\, \partial_\beta(p).$$

recall $\partial_\beta(p) = \frac{\partial}{\partial y^\beta}(u(p))$.

Induced connection on $T^*M \otimes u^{-1}TN$:

$$\nabla(\omega \otimes W) := (\nabla^M \omega) \otimes W + \omega \otimes (\nabla W).$$

(M,g) closed mfld, (N,h) Riem mfld,

$u \in C^\infty(M,N)$.

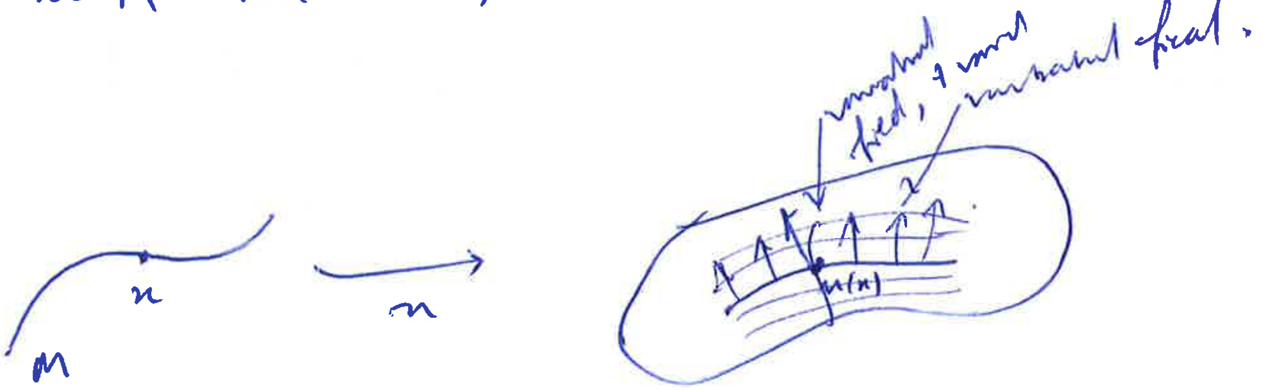
Def^n. $F : M \times (-\varepsilon, \varepsilon) \to N$ C^∞ map is called a variation of u if:

$$F(x,0) = u(x) \quad \forall x \in M$$

Notation: $u_t(x) := F(x,t)$,

Def^n. $V \in \Gamma(u^*TN)$ with $V(x) = \partial_t u_t \big|_{t=0}(x)$ is the variational field of F.

Given $V \in \Gamma(u^*TN)$, define $F(x,t) := \exp_{u(x)}(tV(x))$ $\forall x \in M$ and $t \in (-\varepsilon, \varepsilon)$, $\varepsilon > 0$ suff. small.



Def. Energy of variation field:

$$E[u_t] := \frac{1}{2}\int_M |du_t|^2 \, d\mu.$$

The first variation formula.

Thm. We have that

$$\frac{d}{dt}\Big|_{t=0} E[u_t] = \int_M \langle \nabla V, du \rangle \, d\mu_g$$

Lemma. $F : X \to N$ be smooth and ∇^N L.C. on (N, g). Second connection, defined as

$$\tilde{\nabla}_X\, Y \circ F := \nabla_{dF(X)} Y,$$

i.e. pullback conn. to X on F^*TN.

(I). $\tilde{\nabla}$ compatible with pullback metric on F^*TN

(II). $$\nabla_{dF(e_i)} dF(e_j) - \nabla_{dF(e_j)} dF(e_i) = dF(\underbrace{[e_i, e_j]}) \overset{\shortparallel}{=} [dF(e_i), dF(e_j)].$$

Pf of Thm:

$$\frac{d}{dt}\Big|_{t=0} E[u_t] := \frac{1}{2}\int_M \frac{d}{dt}\Big|_{t=0} |du_t|^2 \, d\mu_g.$$

$$\underset{\text{p.o.u.}\atop\text{indep of } t}{\Rightarrow} = \frac{1}{2}\sum_\alpha \int_{M_\alpha} \varphi_\alpha \sum_{i=1}^m \langle du_t(e_i), du_t(e_i) \rangle \, d\mu_g$$

$$= \sum_\alpha \int_{M_\alpha} \sum_m \langle \nabla_{\partial_t} dF(e_i), du_t(e_i) \rangle \, d\mu.$$

$$= \sum_\alpha \int_{M_\alpha} \sum_m \langle \nabla_{e_i} \underbrace{dF(\partial_t)}_{V \text{ at } t=0}, du_t(e_i) \rangle \Big|_{t=0}$$

$$= \int_M \langle \nabla v, du \rangle \, d\mu_g .$$

Prop. u critical pt iff $\partial_t|_{t=0} E[u_t] = 0$ $\forall v$ variation,

iff $\underbrace{\operatorname{div}(du)}_{\tau(u)} = 0$.

$\overset{\shortparallel}{\tau}(u)$ "Tension field".

Pf. Put $X = \sum_i \langle v, du(e_i) \rangle e_i$, then,

$$\operatorname{div} X = \sum_i \partial_{e_i} \langle v, du(e_i) \rangle .$$

$$= \sum_i \langle \nabla_{e_i} v, du(e_i) \rangle + \langle v, \nabla_{e_i} du(e_i) \rangle .$$

By choosing frame $\longrightarrow$ $= \langle \nabla v, du \rangle + \langle v, \operatorname{div}(du) \rangle$

to be synchronous, ie, $\nabla_{e_i} e_i = 0$ at a point.

Since M closed, $\int_M \operatorname{div} X = 0$

$$\Rightarrow \int_M \langle \nabla v, du \rangle \, d\mu_g = - \int_M \langle v, \operatorname{div}(du) \rangle d\mu_g .$$

Since this holds for all v, ~~$\Rightarrow \operatorname{div}(du) = 0$~~.

Statement follows.

Harmonic maps

(M,g), (N,h) Riem mflds.

$u \in C^\infty(M,N)$, $du \in \Gamma(T^*M \otimes u^*TN)$, $du = du^\alpha \otimes \partial_\alpha$.

$\nabla du \in \Gamma(T^{(2,0)}M \otimes u^*TN)$. Hessian (also 2nd f.f.?).

$\tau(u) = \mathrm{tr}\,\nabla du \in \Gamma(u^*TN)$ Tension field

Def. u harmonic $\Leftrightarrow$ $\tau(u) = 0$.

Remark: M cpt $\Leftrightarrow$ $\tau(u) = 0$.

$$\tau(u) = \mathrm{tr}\,\nabla du = \mathrm{tr}\left(\nabla\left(du^\alpha \otimes \partial_\alpha\right)\right)$$

$$= \mathrm{tr}\left((\nabla du^\alpha) \otimes \partial_\alpha + du^\alpha \otimes \nabla \partial_\alpha\right)$$

$$= \Delta u^\alpha \otimes \partial_\alpha + \nabla_{\mathrm{grad}\, u^\alpha}(\partial_\alpha)$$

$$\cancel{= \Delta u^\alpha \otimes \partial_\alpha +}$$

$$\mathrm{grad}\, u^\alpha = g^{ij}\, \partial_i u^\alpha\, \Gamma^\gamma_{j\alpha}\, \partial_\gamma$$

$$= g^{ij}\, \partial_i u^\gamma\, \partial_j u^\beta \left({}^N\Gamma^\alpha_{\beta\gamma} \circ u\right) \partial_\alpha$$

$$\Rightarrow \tau(u) = \Big(\Delta u^\alpha + \underbrace{g^{ij}\, \partial_i u^\gamma\, \partial_j u^\beta \left(\Gamma^\alpha_{\beta\gamma} \circ u\right)}_{\Gamma(u)(du,du)^\alpha} \Big)\, \partial_\alpha$$

$\Rightarrow$ semi-linear elliptic PDE of order 2.

$\{e_i\}$ local o.n. frame, $\tau(u) = \sum_i (\nabla_{e_i} du)(e_i)$

$$= \sum_i \nabla_{e_i}(du(e_i)) - du(\nabla_{e_i} e_i).$$

2^{nd} variation formula

(M,g) closed, $\quad I = (-\varepsilon, \varepsilon)$, $\quad u : M \to N$ harmonic.

$F : M \times I \times I \to N$, $\quad (x,s,t) \mapsto u_{s,t}(x)$. $\quad$ $\underline{C^\infty\text{-variation}}$.

$\mathcal{I}_e, \quad u_{0,0}(x) = u(x)$.

Variation v.f.: $\quad V(x) = \frac{d}{ds}\Big|_{s=0} F(x,s,0) := dF_{(x,0,0)}(\partial_s)$.

$\qquad\qquad\qquad W(x) = \frac{d}{dt}\Big|_{t=0} F(x,0,t) = dF_{(x,0,0)}(\partial_t)$.

Hessian of E at u: $\quad H(E)_u(V,W) := \partial_s \partial_t \Big|_{t=0} E(u_{s,t})$.

Loc. ONF. $\{e_i\}_i$,

$$E(u_{s,t}) = \frac{1}{2} \int_M \sum_{\alpha,i} \varphi_\alpha \, h(\underbrace{du_{s,t}(e_i)}_{dF(e_i)}, du_{s,t}(e_i)) \, d\mu_g.$$

1st variation formula: $\cancel{\partial_t|_{t=0} E(u_{s,t}) = -\int_M h(dF(\partial_t), \nabla}$

$$\partial_t \cancel{|_{t=0}} E(u_{s,t}) = -\int_M \sum_\alpha \varphi_\alpha h\left(dF(\partial_t), \tilde{\nabla}_{e_i} dF(e_i) - dF(\nabla_{e_i} e_i)\right) d\mu_g.$$

$$\partial_s \partial_t E(u_{s,t}) = -\int_M \sum_\alpha \varphi_\alpha \cdot h\left(\tilde{\nabla}_{\partial_s} dF(\partial_t), \tilde{\nabla}_{e_i} dF(e_i) - dF(\nabla_{e_i} e_i)\right) d\mu_g$$

$$- \int_M \sum_\alpha \varphi_\alpha \cdot h\left(dF(\partial_t), \tilde{\nabla}_{\partial_s} \tilde{\nabla}_{e_i} dF(e_i) - \tilde{\nabla}_{\partial_s} dF(\nabla_{e_i} e_i)\right) d\mu_g$$

$$=: (I) + (II)$$

$\tau(u) = \tilde\nabla_{e_i} dF(e_i) - dF(\nabla_{e_i} e_i) = 0$ sum on harmonic.

$\Rightarrow$ First int grad. $(I) = 0$.

$$\tilde\nabla_{\partial_s} \tilde\nabla_{e_i} dF(e_i) = \tilde\nabla_{e_i} \tilde\nabla_{\partial_s} dF(e_i) + R^N(dF(\partial_s), dF(e_i))dF(e_i).$$

$$= \tilde\nabla_{e_i} \tilde\nabla_{e_i} dF(\partial_s) + dF(\cancel{[\partial_s, e_i]}^{0})$$
$$+ R^N(dF(\partial_s), dF(e_i))dF(e_i).$$

$$\tilde\nabla_{\partial_s} dF(\nabla_{e_i} e_i) = \tilde\nabla_{\nabla_{e_i} e_i} dF(\partial_s) + dF(\cancel{[\partial_s, \nabla_{e_i} e_i]}^{0})$$

At $(s,t) = 0$:

$$\partial_s \partial_t \big|_{(s,t)=0} E[u_{s,t}]$$

$$= - \int_M \varphi_u h\Big(\underbrace{dF_0(\partial_t)}_{w}, \overbrace{\tilde\nabla_{e_i}\tilde\nabla_{e_i} dF_0(\partial_s)}^{\sim} - \tilde\nabla_{\nabla_{e_i} e_i} \underbrace{dF_0(\partial_s)}_{v}\Big) d\mu_g$$

$$+ \int_M {}^N R(\underbrace{dF_0(\partial_s)}_{v}, dF(e_i))dF_0(e_i)\, d\mu_g.$$

$$= \int_M \varphi_u h\Big(-\big(\underbrace{\tilde\nabla_{e_i}\tilde\nabla_{e_i} - \tilde\nabla_{\nabla_{e_i} e_i}}_{\text{rough laplacian}}\big)V$$

rough laplacian $\tilde\nabla^* \tilde\nabla = \Delta_u$ ← depends on u.

$$+ {}^N R(v, du(e_i))du(e_i), w\Big) d\mu_g.$$

$J u = \tilde{\Delta} u - R^N u$. Jacobi operator, & param.

2nd order linear diff op.

$2 \int_M h(\bar{\Delta}_u v, w)\, d\mu_g = \int_M \langle \tilde{\nabla} v, \tilde{\nabla} w \rangle\, d\mu_g$.

Def. $X \in \Gamma(TM)$ s.t. $g(X, Y) = h(\tilde{\nabla}_Y v, w)$. $\forall Y \in \Gamma(TM)$.

$$\Rightarrow \partial_s \partial_t^2 \Big|_{t=0} E[u_{s,t}] = -\int \operatorname{tr} h\left(R^N(v, du(\cdot)) du(\cdot), V\right).$$

$$+ \int_M |\tilde{\nabla} v|^2\, d\mu_g.$$

Examples of Harmonic maps

- (M, g), (N, h) Riem mflds, $u: M \to N$. C^∞.

- $$\begin{array}{ccc} TM, \nabla & TN, \nabla' & u^*TN, {}^u\nabla \\ \downarrow & \downarrow & \downarrow \\ M & N & M. \end{array}$$ ∇, ∇' L.C. ${}^u\nabla$ pullback.

Example 1: (Geodesics).

- $M = I \subset \mathbb{R}$, (N, h) Riem,

$u: I \to N$ smooth curve.

$E(u) = \frac{1}{2} \int_I |du|^2\, dt = \frac{1}{2} \int_I h(\dot{u}, \dot{u})\, dt \Rightarrow u$ geodesic.

$du = e_1^* \otimes \dot{u} \rightsquigarrow. \langle \dot{u}, \dot{u} \rangle \pm$.

$\tau(u) = \mathrm{tr}(\nabla du) = \mathrm{tr}(\nabla(e_i^* \otimes u_i)) = \mathrm{tr}(e_i^* \otimes {}'\nabla u_i) = {}'\nabla_{e_i} u_i$

$\rightsquigarrow$ u geodesic.

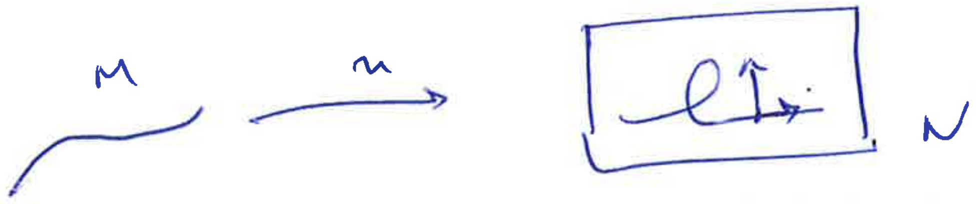
In particular, if $M = S^1$, $u : S^1 \to N$ harmonic $\Leftrightarrow$ u closed geo.

<u>Example 2</u>: Isometric embeddings.

- A smooth map $u : M \to N$ is called an <u>immersion</u> if du_x is injective for all $x \in M$.
- u <u>isometric</u> if $g = u^* h$.

$u^*TN_x = du(T_xM) \oplus du(T_xM)^\perp$

$u^*TN = \mathrm{tan}(u^*TN) \oplus \mathrm{normal}(u^*TN)$. | splitting because du injective.



$TM \underset{du}{\cong} \mathrm{tan}(u^*TN)$ isometric.

$'\nabla$ connection on u^*TN, metric connection.

$du(\nabla_X Y) = \mathrm{tan}({}'\nabla_X du(Y))$. is L.C. connection on TM.

$$\nabla du(X,Y) = {}'\nabla_X du(Y) - \underbrace{du(\nabla_X Y)}_{\mathrm{tan}({}'\nabla_X du(Y))}.$$

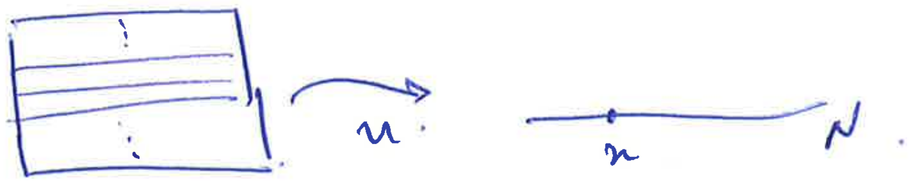
Thus, $\nabla du(X,Y) = \mathrm{nor}({}'\nabla_X du(Y)) = \mathrm{II}(X,Y)$.

So, $\tau(u) = \mathrm{tr}\nabla du = \mathrm{tr}\,\mathrm{II} = m \cdot H$

So u harmonic $\Leftrightarrow$ M minimal.

Example 3. Riem submersions. $u: M \to N$ is a submersion if du_x is surjective $\forall\, x \in M$.

$u^{-1}(u(x))$



M

$$TM = \ker(du) \oplus \ker(du)^{\perp}.$$

$\ker(du) = V$ $(= T\,u^{-1}(u(x)))$. $\ker(du)^{\perp} = H$. $\left(\begin{array}{l} V = \text{vertical} \\ H = \text{horizontal} \end{array}\right)$.

Note: $u^{-1}(u(x))$ is a set, and, in fact, a submanifold.

u is Riemannian sub if

$du_x : H_x \to T_{u(x)}N$ iso

is an isometry. (Rmk. It is always an iso).

- For $X \in \Gamma(TN)$, the v.f. $\tilde{X} \in \Gamma(TM)$ with $\tilde{X}_x \in H_x$ and $du(\tilde{X}) = X$ is called the (unique) horizontal lift.

Claim: A Riem submersion $u: M \to N$ is harmonic $\iff$ the $u^{-1}(u(x))$ are minimal Riem. submanifolds.

Pf. $\{e_i\}$ o.n. frame near $u(x)$, and $\{\tilde{e}_i\}$ their horizontal lifts. I.e., $du(\tilde{e}_i) = e_i$.

Complete $\{\tilde{e}_i\}$ to local o.n. frame.

$\tau(u) = \operatorname{tr} \nabla du = \sum \nabla du(e_i, e_i) = \sum_i \left(\nabla_{e_i} du(e_i) - du(\nabla_{e_i} e_i) \right)$

for $1 \le i \le n$, $\nabla_{e_i} du(e_i) = du(\nabla_{e_i} e_i)$.

$$\Rightarrow \quad \tau(u) = \sum_{i=n+1}^{m} \left(\cancel{\nabla_{e_i} du(e_i)} - du(\nabla_{e_i} e_i) \right).$$

$$= -\sum_{i=n+1}^{m} du(\nabla_{e_i} e_i)$$

$$\tau(u) = 0 \iff \operatorname{hor}\left(\sum_{i=n+1}^{m} \nabla_{e_i} e_i \right) = \operatorname{tr} \mathrm{II}_{u^{-1}(u(x))}.$$

Holomorphic mappings b/w Kähler manifolds.

Prop (Urakawa 3.14):
Holomorphic map $\varphi: M \to N$ where (M, g), (N, h) are Kähler is harmonic.

Def^{n}. $2n$-dim. C^∞ mfld M is called complex if it admits a hol. atlas. That is:

$$z \circ \tilde{z}^{-1} : \tilde{z}(U \cap \tilde{U}) \to z(U \cap \tilde{U}) \quad \text{for any } (z, U), (\tilde{z}, \tilde{U})$$
complex charts is holomorphic.

Def^{n} A map $\varphi: M \to N$ b/w complex manifolds is hol. if $w \circ \varphi \circ z^{-1}$ is hol. for any (z, U) for M, (w, V) for N.

Rem (I). The charts in hol. atlas are holo.

(II). Let $(z = (z^1, \dots, z^m), U)$ and $(w = (w^1, \dots, w^n), V)$.
and $p \in M$, $\varphi(p) \in V$, then $\varphi|_U$ is holo.
iff the Cauchy–Riem eq^{ns} hold.

$$\frac{\partial (u^k \circ \varphi)}{\partial x^j} = \frac{\partial (v^k \circ \varphi)}{\partial y^j} \quad \wedge \quad \frac{\partial (u^k \circ \varphi)}{\partial y^j} = -\frac{\partial (v^k \circ \varphi)}{\partial x^j}.$$

where $z^j = x^j + i y^j$, $w^k = u^k + i v^k$.

M complex mfld, holom. coords, (z, U).
$p \in M$, define complex struc J_p on T_pM, w.r.t. (z, U).
by
$$J_p(\partial_{x^j}) = \partial_{y^j} \wedge J_p(\partial_{y^j}) = -\partial_{x^j}. \quad (*)$$

Rem Def^n is coord indep., $p \mapsto J_p \in \mathrm{End}(TM)$.

Def (I) J sat. $J^2 = -\mathrm{id}$ is called an
<u>almost $\mathbb{C}$. struc</u>.

(II) An <u>almost $\mathbb{C}$ struc</u> J is called $\mathbb{C}$ mfld.
if the struc is defined by charts. $(*)$.

(III) Riem g s.t. $g(JX, JY) = g(X, Y)$.
then (M, g, J) Hermitian manifold.

Def^n M complex, J $\mathbb{C}$ struc, g Herm.. If the
two form $\omega(X, Y) = g(X, JY)$ is closed; i.e. $d\omega = 0$,
then ω is called a Kähler form.

[Ballmann 4.17]: (M,g) Herm. mfld w. ∇ L.C. conn, then:

(I) g kähler, $(d\omega = 0)$.

(II) $\forall p \in M$, $\exists$ local normal hol. charts.

(III) $\nabla J \equiv 0$.

Pf of Prop 3.14. W.T.S $\tau(\varphi) = 0$.

$$\tau(\varphi)(p) = \sum \tau(\varphi)^\alpha \frac{\partial}{\partial \tilde{x}^\alpha}.$$

with $\tau(\varphi)_p^\alpha = -\Delta(\tilde{x}^\alpha \circ \varphi) + \sum \cdots$ ↓ Christoffel symb. w.r.t. the normal coords $= 0$.

Since (N,h) Kähler, Thm (Ballmann):

$$\tau(\varphi)^\alpha(p) = -\Delta(\tilde{x}^\alpha, \varphi) = \sum_{j,k=1}^{2n} g^{jk} \frac{\partial^2}{\partial \tilde{x}^j \partial \tilde{x}^k}(\tilde{x} \circ \varphi).$$

(M,g) Kähler $\Rightarrow$
$$= \sum_{j=1}^{2n} \frac{\partial^2(\tilde{x}^\alpha \circ \varphi)}{\partial \tilde{x}^{j\,2}}$$

$$= \sum_{j=1}^{m} \frac{\partial^2(\tilde{x}^\alpha \circ \varphi)}{\partial x^{j\,2}} + \frac{\partial^2(\tilde{x}^\alpha \circ \varphi)}{\partial y^{j\,2}}.$$

$$= 0$$ $\square$.

Example (I). $\mathbb{C}$ Euclid space is kähler.

(II). $(\mathbb{C}P^m, g_{FS})$ is kähler.

$\to$ Fubini-Study metric.

(III) Hopf fibration $\pi : (S^{2m+1}, g_{S^{2m+1}}) \to (\mathbb{C}P^m, g_{FS})$

Riem. submersion. $\nexists$ kähler kähler.

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Instability theorems

Remarks / Def^{n}s.: $E : C^\infty(M,N) \to \mathbb{R}$.

$\mathrm{Hess}\, E|_\varphi (v,w) = \frac{d^2}{ds\,dt}\Big|_{t,s=0} E(\varphi_{s,t})$.

~~Cla~~ Calculation: $\mathrm{Hess}\, E|_\varphi (v,w) = \int_M h(J_\varphi(v), w)\, d\mu_g$.

$v, w \in \Gamma(\varphi^* TN)$.

Jacobi op. $J_\varphi(v) = -\mathrm{tr}(\nabla\nabla v + R^N(v, d\varphi) d\varphi)$.

Def^n. φ Harmonic map. weakly stable if

$\mathrm{Hess}\, E|_\varphi (v,v) \geq 0 \quad \forall v \in \Gamma(\varphi^* TN)$.

otherwise, unstable. (means $\mathrm{index}(\varphi) > 0$)
weakly stable

$\mathrm{index}(\varphi) = \sup\{\dim F : F \subset \Gamma(\varphi^* TN)$ subspace in which $\mathrm{Hess}\, E|_\varphi$ is neg. def.$\}$.

Note: φ is weakly stable iff $\lambda_i(\varphi) \geq 0$. for eigen.

$J_\varphi v = \lambda v$.

$\mathrm{index}(\varphi) = \sum_{\lambda < 0} \dim V_\lambda(\varphi)$.

Note: $\sec_N \leq 0 \Rightarrow \forall \varphi$ harmonic, φ weakly stable.

$$\int h(J_\varphi(v), v)\, d\mu_g = + \overbrace{\int h(\mathrm{tr}(\nabla\nabla v), v)\, d\mu_g}^{\geq 0} - \underbrace{\int h(\mathrm{tr}(R^N(v, d\varphi) d\varphi, v))}_{\leq 0}$$

$\Rightarrow \geq 0$.

Prep. Vector valued diff forms:

- E v.b., h metric, $\tilde{\nabla}$ connection.
  - $A^r(E) := \Gamma(\Lambda^r T^*M \otimes E)$.
  - $(\tilde{\nabla}_X w)(x_1, \ldots, x_r) = \tilde{\nabla}_X (w(x_1, \ldots, x_r))$

$$- \sum_{r=1}^{m} w(x_1, \ldots, \nabla_X x_i, \ldots, x_r).$$

  - Exterior differential. $r \to r+1$.

$$(d^{\tilde{\nabla}} w)(x_1, \ldots, x_{r+1}) = \sum_{i=1}^{r+1} (-1)^{i+1} (\tilde{\nabla}_{x_i} w)(x_1, \ldots, \hat{x}_i, \ldots, x_{r+1})$$

  - Codifferential: $r+1 \to r$.

$$(\delta^{\tilde{\nabla}} w)(x_1, \ldots, x_r) := \sum_{i=1}^{m} (\tilde{\nabla}_{e_i} w)(e_i, x_1, \ldots, x_r).$$

But: $d^{\tilde{\nabla}} \cdot d^{\tilde{\nabla}} \neq 0$. For a 0-form,

$$(d^{\tilde{\nabla}} d^{\tilde{\nabla}} w)(x, Y) = R^E(x, Y) w.$$

- $\Delta^{\tilde{\nabla}} := d^{\tilde{\nabla}} \delta^{\tilde{\nabla}} + \delta^{\tilde{\nabla}} d^{\tilde{\nabla}} : A^r(E) \to A^r(E)$.
- $\bar{\Delta} w := -\mathrm{tr}(\tilde{\nabla}\tilde{\nabla} w)$.
- $R^{\tilde{\nabla}}(x, Y) w := \tilde{\nabla}_X (\tilde{\nabla}_Y w) - \tilde{\nabla}_Y (\tilde{\nabla}_X w) - \tilde{\nabla}_{[X,Y]} w$.

$A^1(E)$: $(R^{\tilde{\nabla}}(x, Y) w)(z) := R^{\tilde{\nabla}}(x, Y)(w(z))$

$$- w(R(x, Y) z).$$

$$S(w)(x) := \sum_{i=1}^{n} (R^{\tilde{\nabla}}(x, e_i) w)(e_i).$$

Prop Weitzenböck (Bochner) Formula:

$\omega \in A^1(E)$, $\quad \Delta^{\tilde\nabla} \omega = \bar\Delta \omega - \mathcal{S}(\omega)$.

Pf. Let $x_0 \in M$, $\{e_i\}_{i=1}^m$ ONF in nbhd,

$(\nabla_Y e_k)(x_0) = 0$ (synchronous at x_0). $\forall Y \in T_{x_0}M$. $\forall k$.

$$(d^{\tilde\nabla}\delta^{\tilde\nabla}\omega)(x) = (d^{\tilde\nabla}(\delta^{\tilde\nabla}\omega))(x).$$
$$= \tilde\nabla_x \cdot \delta^{\tilde\nabla}\omega.$$
$$= \tilde\nabla_x\left(-\sum_{i=1}^k (\tilde\nabla_{e_i}\omega)(e_i)\right).$$
$$= -\sum_{i=1}^k \tilde\nabla_x\left((\tilde\nabla_{e_i}\omega)(e_i)\right).$$

Similarly,

$$(\delta^{\tilde\nabla} d^{\tilde\nabla}\omega)(x) = (\delta^{\tilde\nabla}(d^{\tilde\nabla}\omega))(x).$$
$$= -\sum_{i=1}^m (\tilde\nabla_{e_i}(d^{\tilde\nabla}\omega))(e_i, x).$$
$$= \sum_{i=1}^m \tilde\nabla_{e_i}(d^{\tilde\nabla}\omega(e_i, x)) - d^{\tilde\nabla}\omega(\underbrace{\nabla_{e_i}e_i}_{=0}, x) - d^{\tilde\nabla}\omega(e_i, \underbrace{\nabla_{e_i}x}_{=0}).$$

synchronicity.

$$= \sum_i \tilde\nabla_{e_i}\left((d^{\tilde\nabla}\omega)(e_i, x)\right).$$
$$= \sum_{i=1}^n \tilde\nabla_{e_i}\left((\tilde\nabla_{e_i}\omega)(x)\right) - (\tilde\nabla_x\omega)(e_i)).$$

On the other hand:

$$(\bar\Delta w)(x) = -\operatorname{tr}(\tilde\nabla\tilde\nabla w)(x).$$

$$= -\sum_{i=1}^{n} (\tilde\nabla\tilde\nabla w)(e_i,e_i)(x).$$

$$= -\sum_{i=1}^{m} [(\tilde\nabla_{e_i}(\tilde\nabla w))(e_i)](x).$$

$$= -\sum_{i=1}^{n} [\tilde\nabla_{e_i}((\tilde\nabla w)(e_i)) - \underbrace{\text{term } \nabla_{e_i}e_i}_{0}].$$

$$= -\sum_{i=1}^{n} (\tilde\nabla_{e_i}\tilde\nabla_{e_i} w)(x).$$

$$= \sum \tilde\nabla_{e_i}((\tilde\nabla_{e_i}w)(x)) + \underbrace{(\tilde\nabla_{e_i}w)(\tilde\nabla_{e_i}x)}_{0}$$

Moreover, $\rho(w)(x) = \sum_{i=1}^{m} (R^{\tilde\nabla}(x,e_i)w)(e_i).$

$$= \sum_i [\tilde\nabla_x(\tilde\nabla_{e_i}w) - \tilde\nabla_{e_i}(\tilde\nabla_x(w)) - \tilde\nabla_{[e_i,x]}w](e_i).$$

$$= \sum_i \tilde\nabla_x((\tilde\nabla_x w)(e_i)) - (\tilde\nabla_{e_i}w)\cancel{(\tilde\nabla_x e_i)} - \tilde\nabla_{e_i}((\tilde\nabla_x w)(e_i)) + (\tilde\nabla_x \cancel{w})(\tilde\nabla_{e_i}e_i) - (\tilde\nabla_{[x,e_i]}\cancel{w})(e_i).$$

$$= \sum_{i=1}^{m} \tilde\nabla_x((\tilde\nabla_{e_i}w)(e_i)) - \sum_{i=1}^{n} \tilde\nabla_{e_i}((\tilde\nabla_x w)(e_i))$$

Prop 1. $\varphi : (M,g) \to (N,h)$. harmonic iff $w := \varphi_* \in A^1(\varphi^* TN)$ harmonic.

i.e., $\Delta^{\tilde\nabla} w = 0$.

Pf $\langle \Delta^{\tilde\nabla} w, w \rangle = |d^{\tilde\nabla} w|^2 + |\delta^{\tilde\nabla} w|^2$.

So, $\Delta^{\tilde\nabla} w = 0$ iff $d^{\tilde\nabla} w = 0$ and $\delta^{\tilde\nabla} w = 0$

Let $X, Y \in \Gamma(TM)$, $d^{\tilde\nabla} w(X,Y) = \tilde\nabla_X (d\varphi(Y)) - \tilde\nabla_Y (d\varphi(X))$
$- d\varphi([X,Y]) = 0$.

Let $\{e_i\}$ o.N.F.
$\delta^{\tilde\nabla} w = -\sum_{i=1}^m (\tilde\nabla_{e_i} w)(e_i) = -\tau(\varphi)$.

and $\delta^{\tilde\nabla} w = 0$ iff $\tau(\varphi) = 0$ ☒

Instability Thm. Let (S^m, g) standard sphere, $m \geq 3$, (N,h) cpt Riem mfld. Then, any nonconstant harmonic map $\varphi : S^m \to N$ is unstable.

Prep for Pf. $S^m := \{x \in \mathbb{R}^{m+1} : \langle x, x \rangle = 1\}$.

$T_x \mathbb{R}^{m+1} = T_x S^m \oplus N_x S^m \cong x^\perp \oplus \mathbb{R} \cdot x$.

$V = \sum_{i=1}^{m+1} a_i \frac{\partial}{\partial x_i} \in T\mathbb{R}^{m+1}$.

$$v(x) = V^T + V^\perp = \underbrace{\sum_{i=1}^{m+1} (a_i - x_i \langle a, x \rangle) \frac{\partial}{\partial x_i}\Big|_x}_{w(x)} + \langle a, x \rangle \sum_{i=1}^{m+1} x_i \frac{\partial}{\partial x_i}$$

Claim 1. $\nabla_x W = -\langle a, x\rangle X$.

Pf. Calculate. $\rightarrow$ due Haar measure.

Claim 2: $\bar{\Delta} W = W$. $\rightarrow$ again calculus.

Key claim $\bar{\Delta}\varphi_* W = \sum_{i=1}^{m} R^N(\varphi_* W, \varphi_* e_i)\varphi_* e_i$
$+ (2-m)\varphi_* W$.

Pf of instability thm:

$$H(E)_\varphi (W,W) = \int_M h\left(\bar{\Delta}\varphi_* W - \sum_{i=1}^{m} R^N(\varphi_* W, \varphi_* e_i)\varphi_* e_i, \varphi_* W\right) d\mu_g$$

$$= \underbrace{(2-m)}_{<0 \text{ as } m \geq 0} \underbrace{\int_M h(\varphi_* W, \varphi_* W)\, d\mu_g}_{\geq 0} \leq 0.$$

If index$(\varphi) > 0 \Rightarrow H(E)_\varphi(W,W) \geq 0 \;\forall W$.

$\Rightarrow \int_M h(\varphi_* W, \varphi_* W)\, d\mu_g = 0 \;\forall W$.

$\Rightarrow \varphi_* W = 0 \;\forall W$.

$\Rightarrow \varphi$ constant. $\lightning$ $\square$.

$\Rightarrow \varphi$ unstable.

Chrisman's interlude

Almost $\mathbb{C}$ – (M, J), $J^2 = -\mathrm{id}$.

$\cup$

Complex. J induced via hol. atlas.

$\cup$

Kähler (M, J, g)

$M = S^6$: S^6 Almost-$\mathbb{C}$:

$S^6 \subset \mathbb{R}^7 = \mathrm{Im}(\mathbb{O})$ $\mathbb{O}$ octonions.

$J_p(x) = x \cdot p$ in ~~quaternions~~ octonions.

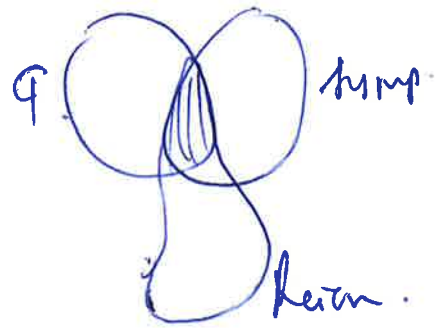
$J_p^2(x) = (x \cdot p) \cdot p = x \cdot (p \cdot p) = -x$.

S^6: Cannot be Kähler

$\omega \rightsquigarrow [\omega] \in H^2(M, \mathbb{R})$
$\neq 0$.

But S^6 has trivial H^2.

Stability and holomorphic maps



Def^{n}. Kähler
- ℂ structure + Herm. metric with ass. 2-form closed
- symp. + integ. almost ℂ structure

  i.e., $g(X,Y) = \omega(X, JY)$
- Riem mfld + parallel almost ℂ structure.

  $g(JX, JY) = g(X,Y)$, $\nabla_X JY = J\nabla_X Y$.

  $(g(X,Y) = \omega(X, JY))$.

Rmk. Ex. projective varieties, stein mfld (ℂ-submflds of $\mathbb{C}^n$).

$E \to M$, M ℂ ~~complex~~ mfld.

If $\exists J: E \to E$ $J^2 = -1 \Leftrightarrow$ structure group $GL(n, \mathbb{C})$.

Holomorphic: E ℂ-mfld, $\pi: E \to M$ hol + local triv. hol.

$\Leftrightarrow$ $g_{uv}: U \cap V \to GL(n, \mathbb{C})$ hol.

Ex. TM for M ℂ-mfld.

$s \in \Gamma(E)$ hol section. if hol. map.

$\Omega^0(E) := \{\text{hol sections}\}$.

$T_p^{\mathbb{C}} M := T_pM \otimes \mathbb{C}$, extend J to $J \otimes \mathbb{C}$: $T_p^{\mathbb{C}}M \to T_p^{\mathbb{C}}M$

$J^2 = -\mathrm{id} \rightsquigarrow x^2 = -1, \Leftrightarrow (x+i)(x-i) = 0$

$T_p^{\mathbb{C}} M = T_p' M$.

$T_p^{\mathbb{C}} M = T_p' M \oplus T_p'' M$. , $T_p' M = \{v \in T_p^{\mathbb{C}} M : Jv = iv\}$
$T_p'' M = \{v \in T_p^{\mathbb{C}} M : Jv = -iv\}$.

$T'M$, $T''M$ hol & anti-hol. tangent bundles.

$T_p M \ni x \longmapsto \hat{x} :=. \frac{1}{2}(x - iJx) \in T_p' M$.

hol. sections of ~~TM~~ $T'M$ are hol. v. fields.

$\partial_{z_j} := \frac{1}{2}(\partial_{x_j} - i\partial_{y_j})$ — span $T_p' M$

$\partial_{\bar{z}_j} := \frac{1}{2}(\partial_{x_j} + i\partial_{y_j})$. — span $T_p'' M$.

$\varphi : M \to N$ holomorphic map, then,

$\varphi^* TN$ hol bundle $\to M$.

"induced hol. tangent bundle."
$\Omega^0(\varphi^* T'N)$ "hol. v.F. along φ".

<u>Main theorems</u>. (M, g) (N, h) compact Kähler, $\varphi : M \to N$ hol.

- φ minimize $E[\cdot]$ in its homotopy class.
- Given a var. of C^1-maps by harmonic maps, i.e,
  $\varphi_t : M \to N$, $\varphi_0 = \varphi$ $t \in (-\varepsilon, \varepsilon)$. is harmonic.
  $(M, g) \to (N, h)$ $\Rightarrow$ φ_t holomorphic.

$\mathrm{Hess}\, E|_\varphi (v,w) = \int_M h(J_\varphi v, w)$ "weakly stable".

$\mathrm{Hess}_\varphi \geq 0$.

<u>If</u>: (M,g), (N,h) are φ-Kähler, $\varphi: M \to N$ hol.

$\rightsquigarrow \int_M h(J_\varphi v, v)\, d\mu_g = \frac{1}{2}\int_M h(Dv, Dv)\, d\mu_g \geq 0$.

for $v \in \Gamma(\varphi^* TN)$. to be defined later.

In particular:

- φ weakly stable, i.e. J_φ has ≥ 0 eigenvalues.
- $\ker J_\varphi = \{ v \in \Gamma(\varphi^* TN) : Dv = 0 \}$.
- $Dv(x) := \tilde{\nabla}_{Jx} v - J \hat{\nabla}_x v$.

<u>"Pair conclusions"</u>:

- v eigenvector of J_φ with eigenval μ:

  $\mu \|v\|_h^2 = \int_M h(J_\varphi v, v)\, d\mu_g \geq 0$.

- $v \in \ker J_\varphi \iff h(J_\varphi v, v) = 0$. Indeed,

  h is +ve def, Jacobi fcn.,

  $h(J_\varphi v, v) = 0 \iff \int_M h(J_\varphi v, v)\, d\mu_g = 0$

  $\iff \int_M h(Dv, Dv)\, d\mu_g = 0 \iff Dv = 0$.

<u>Interpretation</u>:

Bend. th$^=$ is inf dim version of the first

$v \in \Gamma(\varphi^* TN)$ with $Dv = 0$, analytic v. fields along φ.

$\rightsquigarrow \mathcal{A}(\varphi^* TN)$ v.s. of all of them.
$:= \ker(D)$.

Prop. (M,g) (N,h). comp. Kähler; $\varphi: M \to N$ hol.

$$\mathfrak{X}(\varphi^* TN) = \ker(D) \xrightarrow{\cong} \Omega^0(\varphi^* T'N).$$

$$v \longmapsto \tilde{v} := \tfrac{1}{2}(v - iJv).$$

$$(Jv)(p) = J(v(p)).$$

Cor. $\ker J_\varphi = \mathfrak{X}(\varphi^* TN) \cong \Omega^0(\varphi^* T'N)$.

nullity $\varphi := \dim_{\mathbb{R}} \mathcal{J} = \dim_{\mathbb{C}} \mathcal{J}$.

Cor. id on Kähler. is weakly stable.

$\ker J_{id} = \mathfrak{X}(M)$.

$Harm(M,N) = \{$harmonic maps$\}$, $Hol(M,N) = \{$hol maps$\}$

$Hol(M,N) \subset Harm(M,N)$. "submanifold"

"$T_\varphi Hol \subset \mathfrak{X}(\varphi^* TN)$". to be shown.

In show, φ_t variation of $\varphi \in Hol$ in Hol.

$v(p) = \frac{d}{dt}\Big|_{t=0} \varphi_t(p)$ is in $\mathfrak{X}(\varphi^* TN)$. □

(M,g) (N,h) Cpct. , $f \in C^\infty(M,N)$.

Q. Can f be continuously deformed to a harmonic map u?

Ex. $M = S^1 \overset{\text{Sch}}{\Rightarrow} u =$ clsd geo in N.

Th^m. (N,h) non-pos sect. curvature. k_N.

$\Rightarrow \forall f \in C^\infty(M,N)$. $\exists\, u_\infty : M \to N$ harmonic.

s.t. u_∞ homotopic to f.

False if $k_N \neq 0$: $\forall f : \mathbb{T}^2 \to S^2$ of mapping degree ± 1.

Eells - Wood : $\nexists$ harm map : $\mathbb{T}^2 \to S^2$ of mappin deg ± 1.

Approach: Heat-flow method.

Idea : $\mathcal{M} := C^\infty(M,N)$, $u \in \mathcal{M}$ "pt";

$E : \mathcal{M} \to \mathbb{R}$ - functn on $\mathcal{M}$.

Variation. $F. = \{u_t\}_{t \in (-\varepsilon,\varepsilon)}$ curve on $\mathcal{M}$.

Variation-vector $V := \frac{d}{dt} u_t \big|_{t=0} \in \Gamma(u^*TN)$.

$\rightarrow$ also tangent vector $T_u\mathcal{M} \left(:= \Gamma(u^*TN) \right)$.

$\langle\langle w_1, w_2 \rangle\rangle = \int_M \langle w_1, w_2 \rangle_{u^*TN} \, d\mu_g \rightarrow$ inner prod on $\mathcal{M}$.

$$\frac{d}{dt}\Big|_{t=0} E[u_t] = dE_u(V). \text{ - directional derivative.}$$

1st variation formula : $dE_u(v) = -\langle\langle \tau(u), v \rangle\rangle$.

$\Rightarrow \tau(u) = -\mathrm{grad}_u E$.

Deform a given map $u_0 = f \in C^\infty(M,N)$ along grad-flow:

$$\partial_t u_t = \tau(u_t) = (\Delta u_t).$$

For $u: M\times[0,T)\to N$, consider initial val problem.

$$(*)\quad \begin{cases} \partial_t u_t = \tau(u_t). \\ u_0 = f. \end{cases}$$

Looking for sol^{n}s. $u\in C^\infty(M\times(0,T))\cap C^0(M\times[0,T))$. given f.

Q. ① existence!
② when does the sol^{n} converge to harmonic map?

$e(u_t) = \frac{1}{2}|du_t|^2$, $E[u_t] = \int_M e(u_t)\, d\mu_g$.
$k(u_t) = \frac{1}{2}|\partial_t u_t|^2$, $K[u_t] = \int_M k(u_t)\, d\mu_g$.

Prop. Let u solve $(*)$, then:

① $\partial_t e(u_t) = \Delta e(u_t) - |\nabla du_t|^2$ ~~$- \mathrm{tr}_{14}\langle du_t(\mathrm{tr}_{23} Ric^M$~~
$- \mathrm{tr}\langle du_t(\mathrm{tr}_{23} Ric^M(\cdot,\cdot)\cdot), \cdot\rangle.$
$+ \mathrm{tr}\,\mathrm{tr}_{23}\langle R^N(du_t(\cdot), du_t(\cdot))du_t(\cdot), du_t(\cdot)\rangle.$

② $\partial_t k(u_t) = \Delta k(u_t) - |\nabla\partial_t u|^2$
$+ \mathrm{tr}\langle R^N(du_t(\cdot), \partial_t)\partial_t, du_t(\cdot)\rangle.$

Cor. Let u solve $(*)$, $k_N < 0$

① If $\exists\, C>0$ $Ric^M \geq -C\cdot g$,
$\partial_t e(u_t) \leq \Delta e(u_t) + 2C e(u_t).$

② $\partial_t k(u_t) \leq \Delta k(u_t)$.

Prop Let u solve (∞):

① $\partial_t E(u_t) = -2 \cdot k(u_t) \leq 0$.

② $k_N \leq 0 \Rightarrow \partial_t^2 E(u_t) = -2\, \partial_t k(u_t) \geq 0$.

Pf prop

① $\partial_t E(u_t) \overset{\text{1st var}}{=} - \int_M \langle \partial_t u_t, \tau(u_t) \rangle \, d\mu_g$.

$\overset{(\infty)}{=} - \int_M |\partial_t u_t|^2 \, d\mu_g = -2k(u_t) \leq 0$.

(Green).

② $\partial_t k(u_t) = \int_M \partial_t k(u_t) \, d\mu_g \overset{\text{Cor.}}{\leq} \int_M \Delta k(u_t) \, d\mu_g \overset{\text{Green.}}{=} 0$.

Conclusion: $t \to \infty \Rightarrow k(u_t) \to 0 \Rightarrow \partial_t u_t \to 0$.

$e(u) = \frac{1}{2}|du_t|^2 = \frac{1}{2}\langle du_t, du_t\rangle.$

<u>Dokaz</u>: $\Delta e(u_t) = \langle \Delta du_t, du_t\rangle + |\nabla du_t|^2.$

$$\partial_t e(u_t) = \langle \nabla_{\partial_t} du_t, du_t\rangle.$$

$$\Delta du_t = d\,\Delta u_t + \text{curvature}. \qquad \partial_t u_t = \tau(u_t) = \Delta u_t.$$

$$\text{"="} \quad d\,\partial_t u_t + \text{curv}$$

$$\text{"="} \quad \partial_t du_t + \text{curv}.$$

Proof of Prop 4.2

Idea: ptwise calculation in good coordinates – i.e., normal coordinates $\{x^i\}$ at x, $\{y^\alpha\}$ at $u_t(x)$, $t>0$.

Recall
$$\begin{cases} g^{ij}(x) = \delta^{ij}, & \partial_{x^k} g^{ij}(x) = 0 \\ h_{\alpha\beta}(u(x)) = \delta_{\alpha\beta}, & \partial_{y^\alpha} h_{\alpha\beta}(u(x)) = 0 . \end{cases}$$

Also recall
$$\Delta(f_1 f_2) = (\Delta f_1) f_2 + f_1 (\Delta f_2) + g(df_1, df_2).$$

Now, at $(x, u_t(x)) \in M \times N$:
$$\Delta e(u_t) = \frac{1}{2} \Delta \Big(\underbrace{g^{ij} h_{\alpha\beta}\circ u_t}_{\tilde{h}^{ij}_{\alpha\beta}} \, \partial_i u_t^\alpha \, \partial_j u_t^\beta \Big).$$

$$= \frac{1}{2}\left(\Delta \tilde{h}^{ij}_{\alpha\beta}\right) \partial_i u_t^\alpha \partial_j u_t^\beta + \frac{1}{2} \sum_i \sum_\alpha \Delta(\partial_i u_t^\alpha \partial_i u_t^\alpha)$$

$$\left(+ \, g\left(d\tilde{h}^{ij}_{\alpha\beta}, d(\partial_i u_t^\alpha \partial_j u_t^\beta)\right) = 0 \right).$$

$$\frac{1}{2} \sum_i \sum_\alpha \Delta(\partial_i u_t^\alpha \partial_i u_t^\alpha) = \sum_\alpha \sum_i \sum_k \partial_k^2 \partial_i u_t^\alpha \cdot \partial_i u_t^\alpha$$
$$+ \sum_i \sum_\alpha |d(\partial_i u_t^\alpha)|^2$$

$$= \sum_\alpha \sum_i \sum_k \partial_k^2 \partial_i u_t^\alpha \partial_i u_t^\alpha + |\nabla du_t|^2.$$

~~Recall~~ $\cancel{\tau(u_t) = \Big(g^{ij} \partial_i \partial_j u_t^\alpha - g^{ij} \Gamma^k_{ij} \partial_k u_t^\alpha + g^{ij} (\Gamma^\alpha_{\beta\gamma} \circ u_t) \partial_i u_t^\beta \partial_j u_t^\gamma \Big) dx}$

1

Harmonic map ~~equ~~ heat flow equation: $\partial_t u_t^\alpha \partial_\alpha = \tau(u_t)^\alpha$..

~~$\partial_t u_t^\alpha \partial_\alpha = g^{ij}\partial_i\partial_j u_t^\alpha - g^{ij}\Gamma_{ij}^k$~~

~~Differentiate at $(x, u(x)) \in M\times N$ in normal coordinates.~~

$$\partial_t e(u_t) = \frac{1}{2}\partial_t\, g^{ij}(h_{\alpha\beta}\circ u_t)\,\partial_i u_t^\alpha \partial_j u_t^\beta .$$

$$= \sum_i\sum_\alpha \partial_i\partial_t u_t^\alpha\, \partial_i u_t^\alpha .$$

<u>Why?</u> $$\partial_t(h_{\alpha\beta}\circ u_t) = \underbrace{(\partial_{y^\sigma} h_{\alpha\beta})\circ u_t}_{=0 \text{ at } u_t(x)} \cdot \partial_t u_t^\sigma .$$

<u>Relate</u> $\sum_i \partial_i\partial_t u_t^\alpha$ to $\sum_{k,i} \partial_k^2 \partial_i u_t^\alpha$ $\rightsquigarrow$ PDE!

<u>Recall</u>: ~~$\tau(u_t)^\alpha$~~ $\partial_t u_t^\alpha \partial_\alpha = \partial_t u_t = \tau(u_t)$.

$$= \left[g^{ij}\partial_i\partial_j u_t^\alpha - g^{ij}\Gamma_{ij}^k \partial_k u_t^\alpha + g^{ij}(\Gamma^\alpha_{\beta\gamma}\circ u_t)\partial_i u_t^\beta \partial_j u_t^\gamma\right]\partial_\alpha .$$

In normal coords, $\partial_k g^{ij}(x) = 0$, $\Gamma_{ij}^k(x) = (\Gamma^\alpha_{\beta\gamma}\circ u_t)(x)$.

$$\Rightarrow \partial_i\partial_t u_t^\alpha = \sum_j \partial_i\partial_j^2 u_t^\alpha - \sum_j \partial_i\Gamma_{jj}^k \partial_k u_t^\alpha$$
$$+ \sum_j (\partial_{y^\sigma}\Gamma^\alpha_{\beta\gamma})\circ u_t \cdot \partial_i u_t^\sigma\, \partial_j u_t^\beta\, \partial_j u_t^\gamma .$$

2

$$\sum_{i,k,\alpha} \partial_k^2 \partial_i u_t^\alpha \partial_t u_T^\alpha$$

$$= \sum_{i}\sum_{\alpha} \partial_i \partial_t u_t^\alpha \partial_i u_t^\alpha + \sum_{i,j}\sum_{\alpha} \partial_i \Gamma^k_{ij} \partial_k u_t^\alpha \partial_i u_t^\alpha$$

$$- \sum_{i,j}\sum_{\alpha} (\partial_{y^\sigma} \Gamma^\alpha_{\beta\gamma})\circ u_t \, \partial_i u_t^\sigma \partial_j u_t^\beta \partial_j u_t^\gamma \partial_i u_t^\alpha$$

$$= \partial_t e(u_t) + C1 + C2.$$

Remaining term for $\Delta e(u_t)$:

$$\frac{1}{2}(\Delta \tilde{h}^{ij}_{\alpha\beta}) \partial_i u_t^\alpha \partial_j u_t^\beta = \sum_\alpha \frac{1}{2}(\Delta g^{ij}) \partial_i u_t^\alpha \partial_j u_t^\alpha$$

$$+ \sum_i (\Delta h_{\alpha\beta})^{\circ u_t} \partial_i u_t^\alpha \partial_i u_t^\beta$$

$$= L1 + L2.$$

Idea: $L1 + C1 = Ric^M$, $L2 + C2 = Riem^N$.

Some quantities in normal coordinates:

$$\partial_e \Gamma^i_{jk} = \frac{1}{2}(\partial_e \partial_j g_{ki} + \partial_e \partial_k g_{ji} - \partial_e \partial_i g_{jk}).$$

$$Riem_{ikem} = \frac{1}{2}(\partial_k \partial_e g_{im} + \partial_i \partial_m g_{ke} - \partial_k \partial_m g_{ie} - \partial_i \partial_e g_{km}).$$

$$Ric_{ij} = \frac{1}{2}\sum_k (\partial_i \partial_k g_{kj} + \partial_k \partial_j g_{ik} - \partial_i \partial_j g_{kk} - \partial_k^2 g_{ij})$$

$$\Delta g^{ij} = -\Delta g_{ij} = -\sum_k \partial_k^2 g_{ij}$$

$$(\Delta h_{\alpha\beta})\circ u_t = \sum_k (\partial_{y^\sigma} \partial_{y^\gamma} h_{\alpha\beta})\circ u_t \, \partial_k u_t^\sigma \partial_k u_t^\gamma.$$

$(L1) + (C1) = \sum_\alpha \sum_{i,j,k} \left(\partial_i \Gamma^i_{kk} - \frac{1}{2} \partial_k^2 g_{ij} \right) \partial_i u_t^\alpha \partial_j u_t^\alpha .$

$= \sum_\alpha \sum_{i,j,k} \frac{1}{2} \left(\partial_i \partial_k g_{kj} + \partial_i \partial_k g_{jk} - \partial_i \partial_j g_{kk} - \partial_k^2 g_{ij} \right).$

Swap j,i $\quad \partial_i u_t^\alpha \partial_j u_t^\alpha .$

$= \sum_\alpha Ric^k_{ij} \, \partial_i u_t^\alpha \partial_j u_t^\alpha .$

$(L2) + (C2) : -\frac{1}{2} \sum_{ij} \left(\partial_\sigma \partial_\beta h_{\alpha\gamma} + \partial_\sigma \partial_\gamma h_{\beta\alpha} - \partial_\sigma \partial_\alpha h_{\beta\gamma} \right) \circ u_t$
$\partial_i u_t^\alpha \partial_i u_t^\sigma \partial_j u_t^\beta \partial_j u_t^\gamma .$

$+ \frac{1}{2} \sum_{ij} \left(\partial_\sigma \partial_\gamma h_{\alpha\beta} \right) \circ u_t \cdot \partial_i u_t^\alpha \partial_j u_t^\sigma \partial_i u_t^\beta \partial_j u_t^\gamma .$

1. Swap dummy variables $\gamma, \alpha \rightsquigarrow (\partial_\sigma \partial_\alpha h_{\gamma\beta}) \partial_i u_t^\alpha \partial_i u_t^\sigma \partial_i u_t^\beta \partial_i u_t^\gamma$
2. interchange indices $(\alpha,\sigma) \mapsto (\beta,\gamma)$.
3. Chnc. α, σ in 2nd term $\partial_\sigma \partial_\gamma h_{\beta\alpha} \mapsto \partial_\beta \partial_\gamma h_{\alpha\sigma}$.

$\Downarrow$

$= -\frac{1}{2} \sum_{i,j} \left(\partial_\sigma \partial_\beta h_{\alpha\gamma} + \partial_\alpha \partial_\gamma h_{\beta\sigma} - \partial_\sigma \partial_\alpha h_{\beta\gamma} - \partial_\beta \partial_\gamma h_{\alpha\sigma} \right)$
$\partial_i u_t^\alpha \partial_i u_t^\sigma \partial_j u_t^\beta \partial_j u_t^\gamma .$

$= -R^N_{\beta\alpha\sigma\gamma} \, \partial_i u_t^\alpha \partial_i u_t^\sigma \partial_j u_t^\beta \partial_j u_t^\gamma .$

We've met the follow. eq:

$$(\dagger)\quad \begin{cases} \partial_t u_t = \tau(u_t) \\ u(x,0) = f(x). \end{cases}$$

concerning a map $u : M \times [0,T) \to N$, $f : M \to N$.

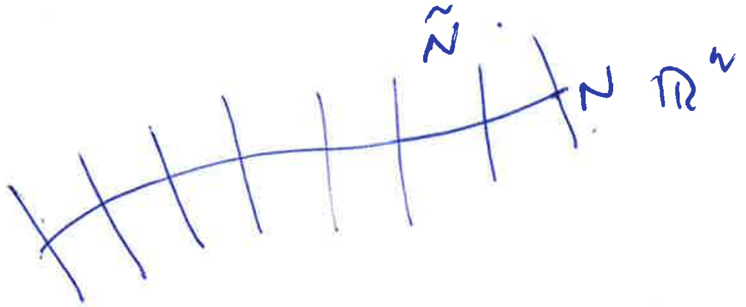
<u>Goal:</u> Show $(\dagger)$ has a sol$^{\underline{n}}$ for small time $T>0$.

"local time dependent sol$^{\underline{n}}$".

<u>Setting</u>. (M,g) is closed Riem.
(N,h). complete Riem. mfld.

<u>Notation</u>: • Let $\iota : N \to \mathbb{R}^q$ denote an isom. emb. into $\mathbb{R}^q$. (Nash).

• Let $\tilde{N}$ denote a tubular nbh of N in $\mathbb{R}^q$.



• $\pi : \tilde{N} \to \iota(N) \subset \mathbb{R}^q$ denote projection.

View map $u : M \times [0,T) \to \tilde{N} \subset \mathbb{R}^q$ as $\mathbb{R}^q$-valued function, consider.

$$(\dagger\dagger)\quad \begin{cases} (\Delta - \partial_t)\, u(x,t) = \Pi(u)\,(du,du)(x,t). \\ \qquad\qquad u(x,0) = \iota \circ f(x). \end{cases}$$

where $\Pi(u)(du,du) = \operatorname{tr}\left(u^*(\nabla d\pi)\right)$.

<u>Objective</u>: Show that a $C^0(M\times[0,T),\tilde{N})\cap \underbrace{C^{2,1}}(M\times(0,T),\tilde{N})$.

2 times cts diff in space
Once cts diff in time.

of (H) correspond to a $C^0(M\times[0,T],N)\cap C^{2,1}(M\times(0,T),N)$.
sol^n of (H).

We need two formulas:

Given. $M_1 \xrightarrow{f_1} M_2 \xrightarrow{f_2} M_3$,

(F1). $\nabla d(f_2\circ f_1) = \nabla df_2(df_1, df_1) + df_2(\nabla df_1)$.
(F2). $\tau(f_2\circ f_1) = \operatorname{tr}\nabla d f_2(df_1,df_1) + df_2(\tau(f_1))$.

<u>Lemma</u> Let u sol^n to (H), with $C^0\cap C^{2,1}$ reg. Then,
$u(M\times[0,T))\subset \iota(N)$.

<u>Pf</u>. Define $\rho:\tilde{N}\longrightarrow \mathbb{R}^q$ by $\rho(z) = z-\pi(z)$, and
let $\varphi: M\times[0,T)\to\mathbb{R}$ by $\varphi(x,t) = |\rho(u(x,t))|^2$
$\rho(z)=0 \iff z\in\iota(N)$. To suffices to show $\varphi\equiv 0$.

Since u satis. (H)

$\partial_t\varphi = 2\langle d\rho(\partial_t u), \rho(u)\rangle = 2\langle d\rho(\Delta u - \Pi(u)(du,du)), \rho(u)\rangle$

$\Delta\varphi = 2\langle \Delta\rho(u), \rho(u)\rangle + 2|\nabla\rho(u)|^2$.

$\Delta\rho(u) = d\rho(\Delta u) + \operatorname{tr}\nabla d\rho(du,du)$ by (F2).

observ: $\pi+\rho = id \Rightarrow d\pi + d\rho = id \Rightarrow \nabla d\pi + \nabla d\rho = 0$.

$\Delta\varphi = 2\langle d\rho(\Delta u) - \operatorname{tr}\nabla d\pi(du,du), \rho(u)\rangle + 2|\nabla\rho(u)|^2$.
$= 2\langle d\rho(\Delta u) - (d\pi + d\rho)\operatorname{tr}\nabla d\pi(du,du), \rho(u)\rangle + 2|\nabla\rho(u)|^2$.

$= 2 \langle d\rho\,(\Delta u - \operatorname{tr} \Pi(u)(du, du)), \rho(u) \rangle$

$\quad - \langle d\pi \operatorname{tr} \nabla d\pi (u)(du, du), \rho(u) \rangle + 2 |\nabla \rho(u)|^2.$

The image of $d\pi$ and ρ are orthogonal.

$\Rightarrow \Delta \varphi = 2 \langle d\rho\,(\Delta u - \Pi(u)(du, du)), \rho(u) \rangle + 2 |\nabla \rho(u)|^2.$

$\Rightarrow \partial_t \varphi = \Delta \varphi - 2 |\nabla \rho(u)|^2;$

Apply div thm:

$$\partial_t \int_M \varphi(\cdot, t)\, d\mu_g = \int_M \partial_t \varphi(\cdot, t)\, d\mu_g$$

$$= -2 \int_M |\nabla \rho(u)|^2 \cdot d\mu_g \leq 0.$$

Given $t \in (0, T)$, have $u_0 \in \iota(N)$,

$$\int_M \varphi(\cdot, t)\, d\mu_g \leq \int_M \varphi(\cdot, 0)\, d\mu_g = 0$$

$\Rightarrow \varphi(u, t) = 0.$ ✓

Prop $u : M \times [0, T) \to N$; set $\tilde{u} = \iota \circ u$.

Sps $\tilde{u}$ is a $C^0(M \times [0, T), \tilde{N}) \cap C^{2,1}(M \times (0, T), \tilde{N})$ sol^n of (H).

Then u is such a sol^n. Also, conversely.

Pf. We have:

① $\Delta \tilde{u} = \operatorname{tr} \nabla d\iota (du, du) + d\iota(\tau(u))$ has F2.

Using $\iota = \Pi \circ \iota$, we also have:

② $\Delta \iota = \operatorname{tr} \nabla d\iota = \operatorname{tr} \nabla d\pi (d\iota, d\iota) + d\pi(\operatorname{tr} \nabla d\iota)$

obs: as $\iota : N \to \mathbb{R}^q$ isom. embed. $\Delta\iota$ is normal to $\iota(N)$

$\Rightarrow d\pi(\Delta \iota) = 0.$

③. So ①+② gives.

$$d\iota(\tau(u)) = \Delta \tilde{u} = \operatorname{tr} \nabla d\tilde{u}\,(d\tilde{u}, d\tilde{u}) = \Delta\tilde{u} - \Pi(\tilde{u})(d\tilde{u}, d\tilde{u}).$$

Since $d\iota(\partial_t u) = \partial_t \tilde{u}$, we have:

$$d\iota(\tau(u) - \partial_t u) = (\Delta - \partial_t)\tilde{u} - \Pi(\tilde{u})(d\tilde{u}, d\tilde{u}). \quad \text{by ③.}$$

$\tilde{u}$ solves (H) $\Rightarrow$ RHS $= 0$.

Therefore LHS $= 0$ as $d\iota$ is injective.

<u>Thm</u>. If a C^2-diff. map $u: M \to N$ satisfies $\tau(u) = 0$, then $u \in C^\infty$.

<u>Pf</u>. Let $\{x^i\}$ be coords and $x \in M$, $\{y^\alpha\}$ and $u(x)$.

Locally, $\Delta u^\alpha = -g^{ij}\Gamma^\alpha_{\beta\gamma}(u)\,\partial_i u^\beta\,\partial_j u^\gamma$.

So, $u \in C^2 \Rightarrow$ RHS is C^1.

$\Rightarrow$ RHS is σ-Hölder cts for $\sigma \in (0,1)$.

by Reg. in diff. soln's to linear elliptic PDEs;

get $u \in C^{2+\sigma}$

$\Rightarrow$ RHS in $C^{1+\sigma}$.

$\Rightarrow u \in C^{3+\sigma}$.

and so on. $\to$ bootstrapping $\Rightarrow u \in C^\infty$

Estimates! Need function spaces.

Let $Q := M \times (0,T)$, $\sigma \in (0,1)$, given v. valued function $u : Q \to \mathbb{R}^d$, let

$$|u|_Q = \sup_{(x,t)\in Q} |u(x,t)|$$

$$\langle u \rangle_x^\sigma = \sup_{\substack{(x,t),\,(x',t')\\ x \neq x'}} \frac{|u(x,t) - u(x',t')|}{d(x,x')^\sigma}.$$

$$\langle u \rangle_t^\sigma = \sup_{\substack{(x,t),\,(x',t')\\ t \neq t'}} \frac{|u(x,t) - u(x',t')|}{|t-t'|}.$$

Define • $|u|_Q^{(\sigma,\sigma/2)} = |u|_Q + \langle u \rangle_x^\sigma + \langle u \rangle_t^{\sigma/2}$.

Rmk $\sigma/2$ – one time derivative, vs $\sigma \to$ two space derivatives.

• $|u|_Q^{(2+\sigma,1+\sigma/2)} := |u|_Q + |D_x u|_Q + |D_x^2 u|_Q^{(\sigma,\sigma/2)} + |D_t u|_Q^{(\sigma,\sigma/2)}$.

~~Rmk $\partial_t u$ need not be diffble~~

We define the fun spaces

$\mathcal{C}^{\sigma,\sigma/2}(Q,\mathbb{R}^q) := \{u \in \mathcal{C}^0(Q,\mathbb{R}^q) : |u|_Q^{\sigma,\sigma/2} < \infty\}$ > B-space

$\mathcal{C}^{2+\sigma,1+\sigma/2}(Q,\mathbb{R}^q) := \{u \in \mathcal{C}^{2,1}(Q,\mathbb{R}^q) : |u|_Q^{2+\sigma,1+\sigma/2}\}$

$\mathcal{C}^{2+\sigma,1+\sigma/2}(Q,N) := \{u \in \mathcal{C}^{2+\sigma,1+\sigma/2}(Q,\mathbb{R}^q) : u(Q) \subset N\}$.

not a linear space, but maybe B-manifold!

Thm. (M,g), (N,h) cpct. Riem. $\forall f \in \mathcal{C}^{2+\sigma}(M,N)$ $\exists\, \varepsilon(M,N,f,\sigma) > 0$ and $u \in \mathcal{C}^{2+\sigma,1+\sigma/2}$ s.t. u is a sol^n of $(+)$.

Strategy: Prove $\exists$ of sol^{n}s of (††) for small time. and use prop to move to sol^{n} of (†).

Thm (Classical) (M,g) Riem. cpct, $Q = M \times [0,T]$

$u: Q \to \mathbb{R}^d$ $\quad Lu = \Delta u + a\nabla u + b\cdot u - \partial_t u$ parabolic PDE.

Consider IVP:

$$\begin{cases} Lu = F(x,t) & (x,t) \in M \times [0,T] \\ u(x,0) = f(x). \end{cases}$$

If the components of $a, b \in C^{\sigma,\sigma/2}$, ~~and~~ $F \in C^{\sigma,\sigma/2}$, $f \in C^{2+\sigma}$, then $\exists$ a unique

$$u \in C^{2+\sigma,1+\sigma/2}(Q, \mathbb{R}^q) \text{ and}$$

$$|u|_Q^{2+\sigma,1+\sigma/2} \leq c\left(|F|^{\sigma,\sigma/2} + |f|^{2+\sigma}\right).$$

Pf of Main thm.

Step 1: Construct approx sol^{n}.

$$\begin{cases} (\Delta - \partial_t)v(x,t) = \pi(f)(df,df)(x). \\ v(x,0) = f(x). \end{cases}$$

$f \in C^{2+\sigma}(M,\mathbb{R}^q) \Rightarrow \pi(f)(df,df) \in C^{\sigma}(M,\mathbb{R}^q)$.

$\overset{\text{classic Thm}}{\Longrightarrow} \exists\, v \in C^{2+\sigma,1+\sigma/2}(M\times[0,T], \mathbb{R}^q)$

~~[illegible]~~ If u_0 sol^n of (#), then. $v(x,0) = u(x,0)$.
and $\partial_t v(x,0) = \partial_t u(x,0)$.

Step 2: Apply Inv. Fun. Th^m:

Fix $0 < \sigma' < \sigma < \Delta$, consider

$$P(u) = \Delta u - \partial_t u - \pi(u)(du, du).$$

A $u \in C^{2+\sigma, 1+\sigma/2}(G, \mathbb{R}^d)$ ~~is the desired sol~~ with $P(u) = 0$ is the desired sol^n. of (#). (with $u(x,0) = f(x)$).

Define. $\mathcal{P}(z) := P(v+z) - P(v)$.

$X := \{ z \in C^{2+\sigma, 1+\sigma/2}(G, \mathbb{R}^q) : z(x,0) = 0,\ \partial_t z(x,0) = 0 \}$.

$Y := \{ u \in C^{\sigma, \sigma/2}(G, \mathbb{R}^n) : u(x,0) = 0 \}$.

$\mathcal{P} : X \to Y$, in phmm, $P(0) = 0$ and is Fréchet diff at 0.

$\mathcal{P} : X \to Y$ Fréchet diff at 0:

$\exists\ P'(0) : X \to Y$ s.t. $\lim\limits_{z \to 0} \| P(z) - P(0) - P'(0)(z) \| \, \|z\|^{-1} = 0$

$$P'(0)(z) = \Delta z - \sum_{i=1}^{q} z^i \partial_i \pi(v)(dv, dv) - 2\pi(v)(dv, dz).$$

$P'(0)$ ~~injective~~ algebraic iso, hold

$\Rightarrow$ open mapp thm $\Rightarrow$ B-space iso!

By IFT, P is an small nbhd of 0 in X and Y.

$\Rightarrow \exists\ \delta > 0$ s.t. $k \in \mathcal{C}^{\sigma,\sigma/2}$ with $k(x,0) = 0$.

and $|u|^{\sigma',\sigma'/2} < \delta$, $\exists\ z \in X$ with $P(z) = k$.

Set $w = \cancel{P}(v) = P(2v) - P(v)$, $u = v + z$,

we can solve $\begin{cases} P(u) = w + k \\ u(x,0) = f(x). \end{cases}$

<u>Step 3</u> (Existence of time local sol$^{\underline{n}}$s).

Consider a C^∞ func $\xi : \mathbb{R} \to \mathbb{R}$ with $\xi = 1$ $(t \leq \varepsilon)$

and $\xi = 0$ $(t \geq 2\varepsilon)$, $|\xi| \leq 1$ $|\xi'(t)| < 2/\varepsilon$.

Then, we can verify that

$$|\xi \cdot w|^{\sigma',\sigma'/2} \leq C\ \varepsilon^{(\sigma-\sigma')/2}\ |w|^{\sigma,\sigma/2}.$$

Set $k = -\xi w$, then $k(x,0) = 0$ and $|k|^{\sigma,\sigma/2} < \delta$.

for sufficiently small ε. $\Rightarrow \exists\ u$ s.t. u sol$^{\underline{n}}$ to

$$\begin{cases} P(u)(x,t) = 0 & (x,t) \in M \times (0,\varepsilon). \\ u(x,0) = f(x). \end{cases}$$

Solve IVP. for small times.

Classical Th$^{\underline{m}}$ $\Rightarrow$ $u \in \mathcal{C}^{2+\sigma,1+\sigma/2}(M \times [0,\varepsilon], \mathbb{R}^q)$

Christian's comment: Avoiding all these kerfuffles with Nash embedding etc.

$$\partial_t u = \tau(u), \quad u(\cdot,0) = u_0 .$$

New operator. $u \mapsto (\partial_t u - \tau(u), u(\cdot,0))$.

$$\mathcal{C}^{\sigma+2,\sigma/2+1}(Q,N) \to \mathcal{C}^{\sigma,\sigma/2}(Q,N) \times \mathcal{C}^{\sigma+2}(M,N).$$

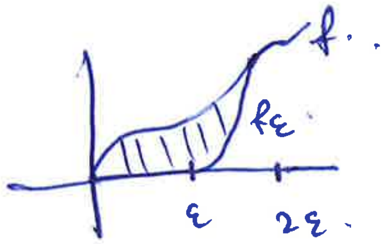
Fix $u \in \mathcal{C}^{\sigma+2,1+\sigma/2}(Q,N)$, linearize at u:

~~Linearize at~~ $$C^{\sigma+2,1+\sigma/2}(Q,u^*TN) \xrightarrow{\text{iso}} \mathcal{C}^{\sigma,\sigma/2}(Q,u^*TN) \times \mathcal{C}^{\sigma+2}(M,u^*TN)$$

$$v \mapsto (\partial_t^\nabla v - J_u(v), v_0).$$

Choose $u(\cdot,0) = u_0$, $\partial_t u(\cdot,0) = \tau(u_0)$.

$\partial_t u - \tau(u) =: f$



For ε small enough, $\exists\, u_\varepsilon$: $\partial_t u_\varepsilon - \tau(u_\varepsilon) = f_\varepsilon$, $u_\varepsilon(\cdot,0) = u_0$

Restrict to time interval $(0,\varepsilon)$ $\to$ the heat equation. Since $f_\varepsilon = 0$.

Long time existence

M, N cpt Riem, $u: M \times [0,T) \to N$, evolving under

$$\partial_t u_t = \tau(u_t), \quad u_0 = f. \quad (*)$$

Thm. $K_N \le 0$, $\mathrm{Ric}_M \ge -C \cdot g$, u sol^n to $(*)$.

- $\partial_t e(u_t) = \partial_t \frac{1}{2}|du_t|^2 \le \Delta e(u_t) + 2 C \cdot e(u_t)$.
- $\partial_t k(u_t) = \partial_t \frac{1}{2}|\partial_t u_t|^2 \le \Delta k(u_t)$.
- $\partial_t E(u_t) = -2 \cdot k(u_t) \le 0$.

Max Principle $f \in C^0(M \times [0,T), \mathbb{R}) \cap C^{2,1}(M \times (0,T), \mathbb{R})$

and $\Delta f - \partial_t f \ge 0$ in $M \times (0,T)$.

$\forall (x,t) \in M \times (0,T)$, $f(x,t) \le \max_{M \times \{0\}} f$.

Pf Pick $\delta > 0$, $t \in [0, T-\delta]$. Show $\forall \varepsilon > 0$

$$\underbrace{f(x,t) - \varepsilon \cdot t}_{=f_\varepsilon} \le \max_{M \times \{0\}} f_\varepsilon.$$

by contradiction.

Sps for contradiction $\exists \varepsilon > 0$, $\exists (x_0, t_0) \in M \times (0, T-\delta]$ s.t.

$f_\varepsilon(x_0,t_0) = \max_{M \times (0,T-\delta]} f_\varepsilon$ not $M \times \{0\}$.

Calculus $\Rightarrow$ $\partial_i f_\varepsilon(x_0,t_0) = 0$, $\partial_t f_\varepsilon(x_0,t_0) \ge 0$, $\mathrm{Hess} f_\varepsilon(x_0,t_0) = (\partial_i \partial_j f_\varepsilon)_{ij} \le 0$

$$\partial_t f_\varepsilon(x_0,t_0) + \varepsilon = \partial_t f(x_0,t_0) \overset{\text{ass.}}{\le} \Delta f(x_0,t_0)$$

$$= \Delta f_\varepsilon(x_0,t_0) = \mathrm{Tr}\,\mathrm{Hess}(f_\varepsilon) \le 0.$$

Rmk. If $\partial_t f \le \Delta f + C f$ for $C \in \mathbb{R}$, then
$f(\cdot,0) \le 0 \Rightarrow f \le 0$.

(Since Pf for $C \le 0$ can always be arranged by
$g(x,t) := e^{-(C+1)t} f(x,t)$.)

Energy estimate: Let $u \in C^0(M \times [0,T), N) \cap C^\infty(\text{interior})$.
Solve $\partial_t u = \tau(u)$, $u_0 = f$.

Ass. $K_N \le 0$, $Ric_M \ge -C \cdot g$.

$\forall\, \varepsilon \in (0,T)$, $\exists\, C^{M,\varepsilon} > 0$:

$$e(u)(x,t) \le C^{M,\varepsilon} e^{2Ct} \cdot E(f). \quad \forall (x,t) \in M \times [\varepsilon, T).$$

Pf. Fix $\varepsilon \le t < T$, $f_1(x,s) := e^{-2C(s+t-\varepsilon)} e(u)(x, s+t-\varepsilon)$.

Then, $\partial_s f_1 \le e^{-2C(s+t-\varepsilon)} \Delta e(u) = \Delta f_1$.

Let $f_2(x,s) := e^{-s\Delta}(f_1(\cdot,0))$, and now

$$\partial_s f_2 = \Delta f_2, \quad f_2(x,0) = f_1(x,0).$$

max principle applied to $f_1 - f_2 \Rightarrow f_1(x,s) \le f_2(x,s)$.

Since $e^{-s\Delta}$ is smoothing for $s > 0$

$$\|f_2(\cdot,\varepsilon)\|_{C^0} \le C^{M,\varepsilon^2} \|f_1(\cdot,0)\|_{L^1} \quad \text{via heat kernel.}$$

$$\le C_1 e^{-2C\cdot(t-\varepsilon)} E(u_{t-\varepsilon})$$

$$\le C_1 e^{-2C(t-\varepsilon)} E(f).$$

$$e^{-2Ct} e(u)(x,t) = f_1(x,\varepsilon) \le f_2(x,\varepsilon) \le C_1 e^{-2C(t-\varepsilon)} E(t). \quad \boxtimes$$

Kinetic estimate : with previous assumption, $\forall (x,t)$

$$|\partial_t u_t(x,t)| \le \max_{M\times\{0\}} |\partial_t u_t|.$$

Pf. $\partial_t k(u_t) \le \Delta(k(u_t)) +$ Max prin.

Hölder estimate for energy

Let $u \in C^0(M\times[0,T),N) \cap C^\infty(\mathrm{int})$, $\partial_t u_t = \tau(u_t)$, $K_N \le 0$.

$\forall \alpha\in(0,1)$ $\exists C>0$ $|u_t|_{C^{2+\alpha}} + |\partial_t u_t|_{C^{\alpha/2}} \le C$.

Pf. Schauder est. for ell. + parabolic PDEs.

$N \overset{\iota}{\hookrightarrow} \mathbb{R}^q$, $\tilde{N} \xrightarrow{\pi} \iota(N) \subset \mathbb{R}^q$. tubular nbh.

$u_t : M \to \mathbb{R}^q$, $\Delta u_t = \pi(u_t)(du_t, du_t) + \partial_t u_t$.

$= \mathrm{Tr}_g\, \mathrm{Hess}(\pi)(du_t, du_t)$.

(a) $|\Delta u_t|_{C^0} \le C_1$

(b) $|u_t|_{C^{1+\alpha}} \le c\,(\underbrace{|\Delta u_t|_{C^0}}_{C^1} + \underbrace{|u_t|_{C^0}}_{C^4}) \le C_5$. (Ell. Schauder)

(c) $|u_t|_{C^{2+\alpha}} + |\partial_t u_t|_{C^{\alpha/2}} \le \dots \le C_6$.

$c\,(|\Delta u_t - \partial_t u_t|_{C^\alpha}) \rightarrow$ where $|du_t|_{C^\alpha} \dots \boxtimes$

Uniqueness theorem: M, N cpt, Riem, let

$u_1, u_2 \in C^0(M\times[0,T), N) \cap C^{2,1}(\text{int})$. solve

$$\partial_t u_i = \tau(u_i) \qquad i = 1, 2.$$

If $u_1|_{M\times\{0\}} = u_2|_{M\times\{0\}} \Rightarrow u_1 = u_2$.

Pf. Let $\varphi(x,t) = |u_1(x,t) - u_2(x,t)|^2$

$$(\Delta - \partial_t)\varphi = 2\langle \Delta(u_1-u_2), u_1-u_2\rangle + 2|du_1 - du_2| - 2\langle \partial_t(u_1-u_2), u_1-u_2\rangle.$$

$$= 2\langle \Pi(u_1)(du_1, du_1) - \Pi(u_2)(du_2, du_2), u_1 - u_2\rangle + 2|du_1-du_2|^2.$$

$$= 2\langle (\Pi(u_1) - \Pi(u_2))(du_1, du_1) + \Pi(u_2)(du_1-du_2, du_1) + \Pi(u_2)\cdot(du_2, du_1-du_2), u_1-u_2\rangle + 2|du_1-du_2|.$$

$$\geq -C|u_1-u_2|\left(|u_1-u_2| + |du_1-du_2|\right) + 2|du_1-du_2|$$

Via Cauchy Schwarz, mean value thm (for $\Pi(u_1)-\Pi(u_2)$):

$$|\Pi(u_1)-\Pi(u_2)(du_1, du_1)| = |\mathrm{Tr}(\mathrm{Hess}_{u_1}(\cdot) - \mathrm{Hess}_{u_2}(\cdot))(du_1, du_1)|$$

$$\leq \|\mathrm{Hess}_{u_1}(\cdot) - \mathrm{Hess}_{u_2}(\cdot)\| \, \|du_1\|^2$$

$$\leq |u_1-u_2| \cdot \sup |\partial_s \mathrm{Hess}_{su_1+(1-s)u_2}|$$

$$\lesssim \|u_1 - u_2\|.$$

Lemma. $\forall a,b \geq 0,\ \varepsilon > 0,\quad ab \leq \varepsilon a^2 + \varepsilon^{-1} b^2.$

Set $\varepsilon = c/2$.

$\Rightarrow (\Delta - \partial_t)\varphi \geq -C\varphi \xRightarrow{\text{max prin}} 0 \leq \varphi \leq 0.$ ☐

Global existence Thm.

$\forall \alpha \in (0,1)$, $f \in C^{2+\alpha}(M,N)$, $\exists!$

$\exists!$ $u \in C^{2+\alpha, 1+\alpha/2}(M \times [0,\infty), N) \cap C^\infty(M \times (0,\infty))$.

s.t. $\partial_t u_t = \tau(u_t)$, $u_0 = f$.

Pf. Let $t_\infty = \sup\{t > 0 : \text{sol}^n \text{ exists on } [0,t)\}$.

Assume $t_\infty < \infty$. By uniqueness.

$u : M \times [0, t_\infty) \xrightarrow{\text{cts}} N$ s.t. $\forall t < t_\infty$,

$u|_{M \times [0,t]} \in C^{2+\alpha, 1+\alpha/2}$.

Let $\alpha' \in (\alpha, 1)$, $t_n \to t_\infty$ $(n \to \infty)$.

Energy estimates yield $\{u_{t_n}\} \subset C^{2+\alpha'}$, $\{\partial_t u_{t_n}\} \subset C^{\alpha'}$

hold uniformly.

$\Rightarrow \exists \{n_k\} \subset \mathbb{N}$: $\begin{cases} u_{t_{n_k}} \to u_\infty \in C^{2+\alpha} \\ \partial_t u_{t_{n_k}} \to \partial_t u_\infty \end{cases}$.

~~t_∞~~
Uniquely determine u extension to t_∞:

$$u(x,t_\infty) = \lim_{t\to t_\infty} u(x,t).$$

This is unique because of Hölder in time.
C^0 is not enough.

Extension $u \in C^{2+\alpha, 1+\alpha/2}(M\times[0,t_\infty], N)$.

Solve, $\partial_t v = \tau(v)$, $v_0 = u(x,t_\infty)$

$\overset{\text{paste}}{\Longrightarrow}$ get $\text{sol}^{\underline{n}}$ on $M\times[0,t_\infty+\varepsilon)$ ↗ . $\boxtimes$

Eells-Sampson Theorem:

Throughout: N C^∞ closed, non pos. curved.
M - C^∞ cpt.

<u>Th$^{\underline{m}}$</u> (ES) In every homotopy class of maps $M\to N$, $\exists$ a harmonic representative.

<u>Prop</u>. Let $f \in C^{2+\alpha}(M,N)$ and $u \in C^{2+\alpha}(M\times[0,\infty),N)\cap C^\infty(\text{int})$ be global time-dep $\text{sol}^{\underline{n}}$ to.

$$\partial_t u = \tau(u), \quad u(0) = f.$$

Then, $\exists$ $\{t_i\}$ $t_i\to\infty$ s.t. $u(\cdot,t_i)\to u_\infty$ s.t. u_∞ harmonic and homotopic to f.

__Pf__. $\left.\begin{array}{l}\{u(\cdot,t)\} \subset C^{2+\alpha}(M,N). \\ \{\partial_t u(\cdot,t)\} \subset C^{\alpha}(M,N).\end{array}\right\}$ bdd, equicts.

$\Rightarrow$ $\exists\, t_i$ s.t. $u(\cdot, t_i) \longrightarrow u_\infty$ uniformly.

$$\partial_t u(\cdot,t_i) = \tau(u(\cdot,t_i)) \longrightarrow \tau(u_\infty).$$

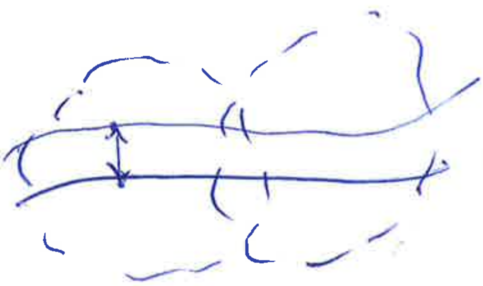
$$\downarrow$$

$$\longrightarrow \partial_t u_\infty = 0.$$

from max's talk, kinetic energy estimate.

__W.T.S__: u_∞ homotopic to f.

Cover N with geo. conv. nbh U_α. For t_i large enough, $u(x,t_i)$ $u_\infty(x)$ lie in same U_α.



~~Since f is a geodesic.~~

~~intersects~~

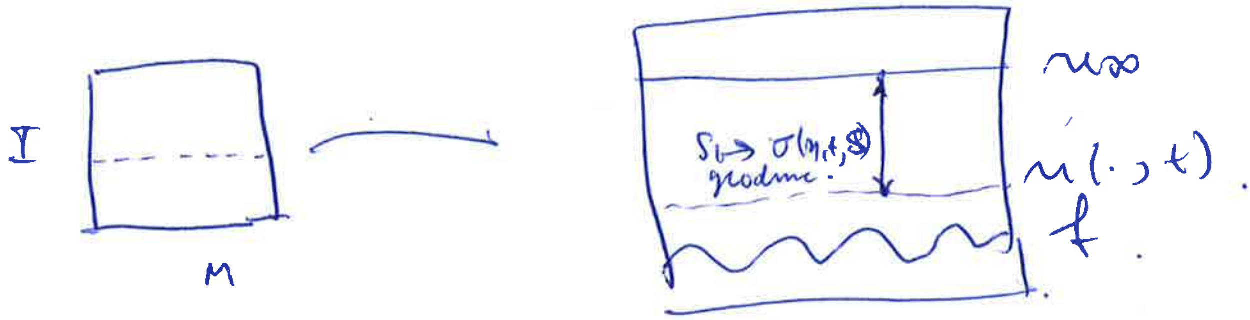
Consider homotopy in each U_α. But intersection is well defined since f is a geodesic so intersection ok. (A)

__Hartman's improvement__: No need for ~~convergence~~ homotopy ~~[illegible]~~ on patches U_α.

~~subsequence~~. Instead show $u(\cdot,t) \longrightarrow u_\infty(\cdot)$ uniformly.

so $u(\cdot,t)$ is the __homotopy__!

Strategy: Show that $d(u(x,t), u_\infty(x))$ is a non increasing function in t.



$$d(u(x,t), u_\infty(x)) = \int_0^1 |\partial_s \sigma(x,t,s)|\, ds.$$

Idea: construct maps $c(x,t,s)$ from $u(x,t)$ to $u_\infty(x)$ s.t. $\theta(t) := \sup_{x\in M} \int_0^1 |\partial_s c(x,t,s)|\, ds$ non increasing.

My Remark: Clearly, the representation is not unique. Given f_1, f_2 two elements in the same homotopy class, $\exists!\, u_\infty^1, u_\infty^2$ with initial f_1, f_2. But, $u_\infty^1 \neq u_\infty^2$ in general. But of course, they are all homotopic to each other.

Steps

Curves c appear as:

Lem. $F \in C^{2+\alpha}(M\times[0,1], N)$ and $u(x,t,s)$ satisfy to:

$$\begin{cases} \partial_t u(x,t,s) = \tau(u(x,t,s)). \\ u(x,0,s) = \sigma(x,s). \end{cases}$$

Then, $V(t,s) = \sup_{x\in M} |\partial_s u(x,t,s)|^2$. and.

$V(t) = \sup_{\substack{x\in M\\ t\in I}} |\partial_s u(x,t,s)|^2$. non-increasing in t.

Pf. $v(x,t,s) := |\partial_s u(x,t,s)|^2$.

Hartman showed - these are C^2.

- $(\Delta - \partial_t) v \geq$ non neg term - curvature term ≥ 0.
- max principle.
- choose $c(x,t,s) = u(x,t,s)$.

Th$^{\underline{m}}$ (Hartman).

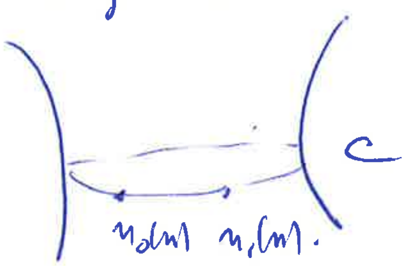
M, N cpt, $\sec_N \leq 0$. Then:

① If u_0, u_1 are harmonic maps homotopic, then $\exists$ homotopy: $H: M\times[0,1] \longrightarrow N$. s.t. for $\forall s\in[0,1]$, $H(\cdot,s)$ is harmonic.

② Let $\sec_N < 0$; Then, if u_0 and u_1 are harmonic homotopic, then $u_0 = u_1$ unless:

(I) u_0 (and hence u_1) is constant.

(II) $u_0(M)$ is a closed geodesic in N and $u_1(M) = c$. Furthermore, $u_1(M)$ is obtained by moving $u_0(M)$ a "fixed oriented distance" along c.



Lemma (Hartman, Thm G).

M, N, and u_0 and u_1 be as in (1). Then, $\exists$ homotopy $f : M \times [0,1] \to N$ s.t. $f(x, \cdot)$ is a geodesic and the length is indep of x and $f(\cdot, s)$ is harmonic.

This proves ① via some subsequential arguments.

Pf of ②:

Let $u_0 \neq u_1$. WTS ~~that~~ $u_0(M) \subset \gamma \leftarrow$ closed geo.

Consider IVP: $\begin{cases} \partial_t u(x,t,s) = \tau(u(x,t,s)), \\ u(x,0,s) = f(x,s). \end{cases}$

$$n(x,t,s) = |\partial_s u(x,t,s)|^2 \not\equiv 0.$$

$$(\Delta - \partial_t) n = |\nabla \partial_s u|^2 - \cancel{\sum \langle R^N(du(e_i)), \partial_s u \rangle}$$

$$- \underbrace{\sum \langle R^N(du(e_i), \partial_s u)\partial_s u, du(e_i) \rangle}_{= 0}.$$

But $\sec_N < 0 \Rightarrow du(e_i)$ and $\partial_s u$ l.d.

$$\Rightarrow \frac{\partial u^\alpha}{\partial x^i} = c^i(x,s) \cdot \frac{\partial u^\alpha}{\partial s}.$$

Diff w.r.t. s, $\frac{\partial c^i}{\partial s} = 0 \Rightarrow c^i$ const. w.r.t. s.

Differentiate w.r.t. $\frac{\partial}{\partial x^j}$:

$$\partial_j \partial_i u^\alpha + \Gamma^\alpha_{\gamma\beta}\, \partial_i u^\gamma \partial_j u^\beta = \partial_j c^i\, \partial_s u^\alpha .$$

$\Rightarrow$ $\partial_j c^i = \partial_i c^j$. LHS symmetric in i, j.

$x_0 \in M$, $x_0 \in U_0 \subset M$, $\exists!\, \varphi : U_0 \to \mathbb{R}$. $\varphi(x_0) \overset{?}{=}$

$$\varphi^i = \partial_i \varphi = -c^i .$$

$$0 = \partial_i u^\alpha (x, \varphi(x)+s) + c^i(x)\, \partial_s u^\alpha (x, \varphi(x)+s) .$$

$= \partial_i u^\alpha (x, \varphi(x)+s)$. $\Rightarrow$. $u(x, \varphi(x)+s) = u(x_0, \varphi(x_0)+s)$

$\Rightarrow$ c^i $\underline{\underline{\text{const.}}}$ min.

$$\left.\begin{array}{l} \gamma_{x_0} \\ \gamma_x \end{array}\right\} \overset{p98}{\Rightarrow} \gamma_{x_0} = \gamma_x .$$

M connected $\Rightarrow$. $u_0(M) \subset \gamma_{x_0}$.

Finishing Pf. Let u_0 be a constant map. Want to show u_1 cont. map.

$$\gamma_u$$

$$u_1(x_1).\quad u_1(x_2).$$

Let $u_0 \neq$ const. u_0 not be closed $\Rightarrow$ u_0 is unclear

Th^m of Preissmann and other apps.

Th^m. (M,g), (N,h) cpct Riem., $\sec_N \le 0$. $\Rightarrow$ $\forall f \in C^0(M,N)$. homo. to harm. map u_∞.

Th^m $\sec_N < 0$ and u_0, u_1, harm. maps homotopic, then:
(I) u_0 and therefore u_1 const maps.
(II). $u_0(M)$ closed geodesic in N and $u_1(M) = C$.

Prop. $u : M \to N$ harmonic, then:

$$\Delta e(u) = |\nabla du|^2 + \langle du \circ ric_M, du \rangle_{u^*TN} - tr.tr_{23} \langle R^N(du_+(\cdot), du_+(\cdot))du_+(\cdot), du_+(\cdot) \rangle.$$

Weitzenböck formula $e_1, \dots, e_m$ is a local ~~frame~~ frame.

Pf. from Weitzenböck formula for harmonic map heat flow:
If u harmonic, $u_t = u$ sol^n to $\partial_t u_t = \tau(u_t)$, $u_0 = u$.
and $\partial_t e(u_t) = 0$.

Cor. M cpct connected Rien, non-neg $Ric_M \ge 0$, N Riem, cpct, $\sec_N \le 0$,
$u : M \to N$ harmonic:
1. u is totally geo ($\nabla du \equiv 0$), $e(u) \equiv$ const.
2. $Ric^M \not\equiv 0 \Rightarrow u$ constant. ($Ric^M \not\equiv 0 \Leftrightarrow \exists x$ s.t. $Ric^M_x > 0$).
3. $\sec_N < 0 \Rightarrow u$ is either constant or maps M into a closed geodesic of N.

Pf 1. $\int_M \Delta e(u) \cdot d\mu_g = 0$. by ~~div thm~~ ~~the first term are harmonic~~ divergence theorem.
curv. holds $\Rightarrow$ $\nabla du = 0 \Rightarrow \Delta e(u) = 0 \Rightarrow e(u) \in \ker(\Delta) = \{\text{const}\}$.

2. If $\exists x$: $Ric^M_x > 0$, $\langle du \circ ric^M, du \rangle \equiv 0 \Rightarrow du_x = 0$.
$e(u)_x = \frac{1}{2}|du_x|^2 = 0 \Rightarrow e(u)$ is const. $\Rightarrow e(u) = 0$.

3. Assumption means that $du(e_i), du(e_j)$ can not be linearly indep.
$\Rightarrow \dim u(M) \le 1$.

$\dim u(M) = 0 \Rightarrow u$ constant, otherwise $u(M)$ closed geodesic.

$\nabla du \equiv 0 \Leftrightarrow u$ maps geodesics of M to geodesics of N.

$\quad | \quad \nabla_{\partial_t}(u\circ\gamma)'(t) = \cdots = \nabla du(\gamma', \gamma')$.

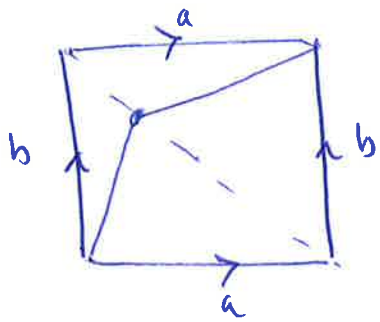
Thm (Preissmann). M Riem cpt. $Sec_M < 0$. Then, every abelian subgroup of $\pi_1(M)$ is infinite cyclic ($\cong \mathbb{Z}$).

Pf. Let $a, b \in \pi_1(M, x_0)$, assume they commute.

$\Rightarrow \exists$ homotopy b/w $a\cdot b$, $b\cdot a : [0,1] \to M$

$f : [0,1]\times[0,1] \to M$, $f(0,s) = f(1,s)$ $\forall s \in [0,1]$.

$\Rightarrow$ induces map $f : \mathbb{T}^2 \to M$.



Using flat structure on $\mathbb{T}^2 = S^1\times S^1$, $\exists$ harmonic map $u : \mathbb{T}^2 \to M$ homotopic to f.

Curvature hyp $Sec_M < 0 \Rightarrow u(T^2)$ either constant or a closed geodesic in M, say c. Then c has base pt. $x_1 = u(0,0)$.

$\Rightarrow [\tilde a] = [c]^k$, $[\tilde b] = [c]^m$. $\Rightarrow \tilde a, \tilde b$ contained in the cyclic subgroup generated by c.

$\tilde a = u(0,\cdot)$. $\quad \tilde b = u(\cdot, 0)$.

$\Rightarrow$ this group has to be infinite, otherwise, $[c]^k = e$, which contradicts that $u \equiv$ const or $u_1(T^2)$ geodesic.

$\Rightarrow [c]^k$ is geodesic because c is geodesic and e is trivial geodesic.

$\Rightarrow \tilde a \tilde b$ and $\tilde b \tilde a$ must be contained in a inf. cyclic subgroup of $\pi_1(M, x_1)$.

$\pi_1(M, x_1) \cong \pi_1(M, x_0)$ $\quad \tilde a \mapsto a$, $\tilde b \mapsto b$.

M Riem mfld $\Rightarrow$ G of isometries of M is a Lie group.
M cpt $\Rightarrow$ G cpt.

Thm. M connected, cpt. with $\mathrm{Ric}_M < 0 \Rightarrow G$ is finite.

Pf ① $f \in G$. homotopic to id $\Rightarrow f = id$.

f is harmonic map since isometries with $\nabla df = 0$.

Thm of Hartman: $\mathrm{Ric}_M < 0 \Rightarrow$ uniqueness of $f \Rightarrow f = id$.

~~G is discrete~~

WLOG, ~~sps then~~ G is discrete: ~~take~~ (G) id isolated pt in, Sps it is not, and V a nbhd of id.

G Lie group, V diffeo to $\mathcal{U}$.

$n \in \mathcal{U}$, $\exists\ X \in \mathfrak{g}$: $n = \exp(X)$, we have homotopy

$g(t,n) := \exp(tX)(n)$. connects n and id.

holds for all $n \in \mathcal{U} \Rightarrow V = \{id\}$.

$\square$.

G discrete + cpt $\Rightarrow$ finite.

Thm. M complex submanifold $\subset$ Kähler. N

$\Rightarrow$ M is minimal. ($H = \mathrm{tr}\, \mathrm{II} = 0$).

Pf M C submfld. $\exists\ u: M \to N$ analytic embedding.

M itself Kähler with induced metric, and $u: M \to N$ harmonic.

and $\mathrm{tr}\nabla du = 0$, here $\mathrm{tr}\nabla du = \mathrm{tr}\,\mathrm{II}$.

$\square$

2-dim Harm maps.

<u>Convention</u>: $\Sigma_1, \Sigma_2, \Sigma_3$ Riem surf., N Riem mfld (no dim restr)

$\partial_z := \frac{1}{2}(\partial_x - i\partial_y)$, $\partial_{\bar z} := \frac{1}{2}(\partial_x + i\partial_y)$.

<u>Def</u>: $\langle \cdot, \cdot \rangle_\Sigma$ conformal ~~if in local coords~~ $\exists\, \rho : \mathbb{C} \to \mathbb{R}_+$ s.t.

$\langle \cdot, \cdot \rangle_\Sigma = \rho^2(z)\, dz \otimes d\bar z$.

$f : \Sigma \xrightarrow{C^1} N$ <u>conformal</u> $\iff$ $\langle \partial_z f, \partial_z f \rangle_N = 0$.

<u>Note</u>: this does <u>not</u> imply $\partial_z f = 0$.

<u>Rmk</u>. $f : \Sigma_1 \to \Sigma_2$ (anti-) holom., $\langle \cdot, \cdot \rangle_{\Sigma_2}$ conformal $\Rightarrow$ f conformal.

<u>Lem</u>. $\langle \cdot, \cdot \rangle_\Sigma$ conformal. (with λ). Then, $\Delta = \dfrac{-4}{\lambda^2(z)} \partial_z \partial_{\bar z}$.

f is harmonic iff. $\partial_z \partial_{\bar z} f^i + \Gamma^i_{jk}(f(z)) \cdot \partial_z f^j \partial_{\bar z} f^k = 0$.

<u>Rmk</u>: • indep of the choice of ~~conformal~~ conf. metric.

- $f : \Sigma_1 \to \Sigma_2$ (anti-) holo. and f harmonic
- $k : \Sigma_1 \to \Sigma_2$ (anti-) holo, $f : \Sigma_2 \to N$ harmonic.

$\Rightarrow f \circ k$ harmonic. $\leftarrow$ NOT TRUE in gen, very special to 2-dim.

- $E[f : \Sigma \to N] := \int_\Sigma g_{ij}\, \partial_{\bar z} f^i \partial_z f^j \, dz \wedge d\bar z$ indep. of $\langle \cdot, \cdot \rangle_\Sigma$.

and $E[f \circ k] = E[f]$. for $f : \Sigma \xrightarrow{C^1} N$, $k : \Sigma_1 \cong \Sigma_2$.

bi (anti-) holo.

Prop. $f: \Sigma \to N$ harmonic, then $\underbrace{\langle \partial_z f, \partial_z f\rangle_N}_{=\,\Psi(z)} dz^2$

is a holom. quad. diff. (i.e., $\in \Gamma(T^*_{\mathbb{C}}\Sigma^{\otimes 2})$) and $\Psi(z)dz^2 = 0$ iff f conformal.

Rmk $\Psi(z)dz^2 = (f^* g_N)_{Hol}$. so global. $(\cdot)_{Hol}$ hol projection.

Remark: $\mathbb{C}P^1 = S^2$ with charts.

$$S^2 \setminus \{(0,0,\pm 1)\} \to \mathbb{C}, \quad (x_1,x_2,x_3) \mapsto \frac{1}{1\pm x_3}(x_1 \pm i x_2).$$

Lem Every hol. quad. diff on S^2 vanishes. In particular, every harmonic $h: S^2 \to N$ is conformal.

Pf Restrict $\varphi: \mathbb{C} \to \mathbb{C}$ holom, $z = f_1(x)$, $w = f_2(x)$ $(= \frac{1}{z})$.

$z \neq 0$, $\varphi(z)dz^2 = \varphi(z(w))\left(\frac{\partial z}{\partial w}\right)^2 dw = \varphi(z(w))\frac{1}{w^4}dw^2$ hol

So, $\varphi(z) \to 0$ as $z \to \infty$. $\Rightarrow \varphi \equiv 0$. (Liouville).

$h_\lambda: S^2 \to S^2$, $z \mapsto \lambda z$, for $\lambda \in \mathbb{C}^\times$, $E(h_\lambda) = E(h_\lambda \circ h_2) = E(h_1)$ $\in \mathbb{C}^\times$.

$\rightarrow$ sequence of harmonic maps with $E[h_\lambda] = \text{const} \neq 0$.

converges for $\lambda \to 0$ pointwise almost everywhere to a $h_0 = 0$ with $E(h_0) \neq \lim_{\lambda \to 0} E(h_\lambda)$.

Def. Σ Riem. surf. w/ bdry.

$A = \{\varphi : U \to V \subset \mathbb{H} : \varphi_1 \circ \varphi_2 \text{ holo}\}$.

Ex. $D^2 = \{z \in \mathbb{C} : |z| \leq 1\}$.

$A = \{ \cdot \ \text{—} \ \cdot \}$ ← can't read warning.

Def. $q \in \Omega^0(T^{*}_{\mathbb{C}}\Sigma \otimes T^{*}_{\mathbb{C}}\Sigma)$ hol. diff called real if $\forall z_0 \in \partial\Sigma$, $\forall v \in T\partial\Sigma$ $f(v,v) \in \mathbb{R}$.

In local coords: q "=" $\varphi(z)\, dz^2$,

q real iff $(\operatorname{Im} \varphi)(z_0) = 0$.

Lem. q hol. diff on D^2 real $\Rightarrow$ $q \equiv 0$.

Pf. Define explan. of $\varphi\, ((\text{Cayley})^{-1})^{*} q = \varphi(z)\, dz^2$.

$\operatorname{Im} \varphi|_{\mathbb{R}} = 0$.

and now it cross. bdry axis via $\tilde{\varphi}(z)\, d\tilde{z}^2 = \begin{cases} \varphi(z) & \operatorname{Im} z \geq 0 \\ \varphi(\bar{z}) & \operatorname{Im} z \leq 0. \end{cases}$

Get $\Phi(z) = \tilde{\varphi}\, dz^2$ to be $\Phi \equiv 0 \Rightarrow q \equiv 0$.

Thm. $h : D^2 \to (N, g_N)$, $h|_{\partial D^2} = \text{const} \Rightarrow h = \text{const on } D^2$.

Lemma (Hartman–Wintner)

$0 \in \Omega \subset \mathbb{C}$, $u \in C^2(\Omega, \mathbb{R}^d)$ with $|u_{z\bar{z}}| \leq k(\Omega)\,(\ldots)$

too hard to read.

Consequence: $\Sigma \to N$ non-const. harm. maps. satisfies an "identity th." ? i.e., pre-images of pts discrete.

Existence of harm maps in dim 2.

Th$^{\underline{m}}$ 1 Σ cpt Riem surf., N cpt Riem., $\pi_2(N) = 0$.
Then, any smooth map $\varphi : \Sigma \to N$ is homo. to harm map.

Lemma (Retraction Lemma)
N Riem mfld, $B_0 \subset B_1 \subset N$, B_i closed, $\pi : B_1 \to B_0$ C^1 with
$\pi|_{B_0} = id_{B_0}$, $\|D\pi(v)\| \le \|v\|$ $\forall x \in B_1 \setminus B_1$ $v \in T_xN, v \neq 0$.
Let M be Riem w/ bdry ∂M and $h : C^0 \cap W^{1,2}(M, B_1)$,
$h(\partial M) \subset B_0$, h energy minimizing in the class of such maps w/ same bdry values. $\Rightarrow h(M) \subset B_0$.

Lem (Courant-Lebesgue lemma).
N Riem, $d_N(\cdot,\cdot)$ distance, $u \in W^{1,2}(D, N)$. $E[u] \le k$.
Then, $\forall x_0 \in D$, $\forall \delta \in (0,1)$, $\exists$ $\rho \in (\delta, \sqrt{\delta})$ $\forall x_1, x_2 \in D$
$|x_i - x_0| = \rho$ s.t.

$$d(u(x_1); u(x_2)) \le \frac{(8\pi k)^{\frac{1}{2}}}{(\log \frac{1}{\delta})^{\frac{1}{2}}}$$

Intermedi Th$^{\underline{m}}$ N complete Riem, $\sec_N \le k$, $k > 0$, inj rad $i_0 > 0$.
~~PEN~~ Let $r \in (0, \min(i_0/2, \frac{\pi}{2\sqrt{k}}))$. Fix $p \in M$,
Sps $g \in C^0 \cap W^{1,2}(\partial D, B(p,r))$, s.t. $\bar{g} \in H^{1,2}(D, B(p,r))$ extension.
Then, $\exists$ harmonic $h : D \to B(p,r)$ w/ $h|_{\partial D} = g$.
On $\{z : |z| \le 1 - \sigma\}$, $\forall \sigma > 0$, mod. cts. depends only on $\sigma, r, k, E[\bar{g}]$.

Reg. in dim 2

1. overview: M, N Riem, Σ Riem surface.

Def. $\iota : N \to \mathbb{R}^q$ isometric embedding.

$f \in H^{1,2}_{loc}(M,N) \Leftrightarrow \forall \varphi$ charts, $\iota \circ f \circ \varphi^{-1} \in H^{1,2}(M, \mathbb{R}^q)$.

$f \in H^{1,2}(M,N) \Leftrightarrow f \in H^{1,2}_{loc}(M,N)$. and $E(f) := \frac{1}{2}\int_M |df|^2 < \infty$.

and $\underline{f \in L^2(M,N)}$.

Def^n $u \in H^{1,2}(M,N)$ weakly harmonic $\Leftrightarrow$ u critical pt. of E.

$\Leftrightarrow$ $\forall$ bdd + cptly spt. sections $\varphi \in u^*TN$, $u_t := M \to N$.

variation of u (say, has $u_t(p) := \exp_{u(p)}(t\varphi(p))$,

one has. $\frac{d}{dt}\big|_{t=0} E[u_t] = 0$.

$\Leftrightarrow$ $\forall \varphi_{\in u^*TN}$ satisfying ~~..~~ spt φ cpt $\int_M \langle df, \nabla\varphi \rangle = 0$.

Th^m. 1 (Ladyzhenskaya - Ural'tseva 68):

$f \in H^{1,2}(M,N) \cap C^0$ harmonic map $\Rightarrow$ $f \in C^\infty$.

Ex. $h : \mathbb{R}^3 \to S^{n-1}$, $h(x) = \frac{x}{|x|}$ weakly harmonic if

$n \geq 3$. But $\underline{\underline{\text{not}}}$ harmonic!

Th^m. 3 (Hélein, 1991): N cpct, $h : \Sigma \to N$ weakly harm.

$\Rightarrow h \in C^0(\Sigma, N)$.

Corollary 4: N cpct, $h : \Sigma \setminus \{p\} \to \underline{\text{harmonic}}$, then Σ extends

to harmonic map $\Sigma \to N$.

$E[h] < \infty$
not sufficient los

2. What is better in dim 2?

<u>Lem</u> Let $h \in H^{1,2}(\Sigma, N)$. Then, h weakly harmonic.

$\iff$ $\forall$ hol. charts φ of Σ, $h \circ \varphi^{-1} : U \subset \mathbb{C} \to N$ weakly harm.

<u>Lem.</u> (Courant-Lebesgue).
If $f \in H^{1,2}(D^2, N)$, $\forall \varepsilon > 0$ $\exists$ $\delta \in (0, \varepsilon)$ s.t. $\operatorname{osc}(f|_{\partial D^2_\delta}) < \varepsilon$.

<u>Thm.</u> (Grüter 81):

- N has $\operatorname{inj}(N) \geq i_0 > 0$, $|\mathrm{Sec}| \leq 1$.
- $h : \Sigma \to N$ weakly harm + conf. a.e.

$\Rightarrow$ $h \in C^0(\Sigma, N)$.

More happened, but it's too late, and I stopped taking notes.

obs. $\langle A_{u}, u\rangle = \langle\langle A, u\circ u\rangle\rangle$. $\quad u\circ u(v) = \langle u, v\rangle u$.

Def. $W_0 = \mathrm{span}\{x(v_0\circ v_0) : x \in G\} \subset S$.

$E_1 = W_0^{\perp} \in S$.

$L_1 = \{C \in E_1 : C + 1 \geq 0\}$.

Prop L_1 cpct & linear.

Def^n. (I) $\varphi, \psi : M \to \mathbb{R}^{n+1}$ "maps equivalent" $\Leftrightarrow \exists\, \tau \in O(n+1),\ \psi = \tau \circ \varphi$.

(II) $\varphi : M \to S^n$ is full $\Leftrightarrow \varphi(M)$ not contained in an $(n-1)$ dim. hypersurface of S^n.

Th^m. (do Carmo, Wallach)

(M, g) cpct. hom. space, Riem. Then:

(I) If $\varphi : M \to S^n$ is full eigenmapping with $e(\varphi) = 1/2$.

$\Rightarrow \lambda \in \mathrm{spec}(\Delta_g)$ and $n \leq n_1$.

(II). $L_1 \to \mathcal{E}_1(M) := \{$full eigen. mappings $M \to S^n\}/\sim$. $e(\varphi) = 1/2$.

$C \mapsto [(C+1)^{1/2} \circ \varphi_1]$. bijection.

- $\mathring{L}_1$ corresponds to full eigenmappings. $M \to S^{n_1}$.
- ∂L_1 " " " $M \to S^n$. $n < n_1$.

Application: Th^m: the only embedded minimal torus in S^3 with the flat metric is the clifford torus.

Preparations:

① $M = G/K$, G cpt. Lie group $K \subset G$ closed subgroup.

- $G \curvearrowright G/K$, metric g G-invariant.
- $G \times C^\infty(M) \longrightarrow C^\infty(M)$, $(x\cdot f)\,g(n) = f(x^{-1}g\,n)$.

② $\lambda \in \mathrm{spec}(M) := \mathrm{Spec}(\Delta_g)$,

- $V_\lambda := \mathrm{Eig}(\Delta_g, \lambda)$, $n(\lambda) := \dim(V_\lambda) - 1$.
- For $f_1, f_2 \in V_\lambda$, $\langle f_1, f_2 \rangle = \dfrac{n_\lambda + 1}{\mathrm{vol}(M)} \int_M f_1 f_2 \, d\mathrm{vol}$.

<u>Lemma 2</u> a) V_λ are G-invariant, b) $\langle \cdot, \cdot \rangle$ is G-invariant.

③ $\{f_\lambda^i\}$ o.n.b for V_λ,

$\overline{\varphi}_\lambda : M \longrightarrow V_\lambda \cong \mathbb{R}^{n_\lambda+1}$, $\overline{\varphi}_\lambda(xK) = \sum f_\lambda^i(xK)\, f_\lambda^i \cong \begin{pmatrix} f_\lambda^1(xK) \\ \vdots \\ f_\lambda^{n+1}(xK) \end{pmatrix}$.

<u>Lem 3</u>. $\overline{\varphi}_\lambda(xK) \in S^{n_\lambda}$.

<u>Pf</u>. Calculation and the integral, $\langle \cdot, \cdot \rangle$ condition. $\square$.

<u>Rem</u>: a) $\overline{\varphi}_\lambda$ induces standard eigen map $\varphi_\lambda : M \to S^{n_\lambda}$.

b) For $A \in O(n_\lambda + 1)$ s.t. $A\varphi_\lambda(n) = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \theta$.

$\rightsquigarrow$ new o.n.b. $f_\lambda'^i$,

$\rightsquigarrow$ new eigenmap φ_λ' s.t. $\varphi_\lambda' = A \circ \varphi_\lambda$.

<u>Idea</u>: use $A \mapsto A \circ \varphi_\lambda'$ to parametrize $\{\text{eigen maps}\}$.

<u>Def</u>: $\mathcal{S} := \{ A \in \mathrm{End}(V_\lambda) : \langle Au, v \rangle = \langle u, Av \rangle \}$.

$\langle\langle A_1, A_2 \rangle\rangle = \mathrm{tr}(A_1 A_2)$. G-inv.

G-action on $\mathcal{S}$: $(x\cdot A)(u) = x(A^{-1}(x^{-1}\cdot u))$.

Rks by Takahashi & do Carmo/Wallach.

Goal: Study harmonic maps $\underset{\text{Cpt Riem}}{(M,g)} \xrightarrow{\varphi} (S^n, g_{round}) \overset{\iota}{\hookrightarrow} (\mathbb{R}^{n+1}, g_{euc})$.

$\Phi := \iota\circ\varphi = \begin{pmatrix}\Phi_1\\ \vdots \\ \Phi_{n+1}\end{pmatrix}$.

Thm (Takahashi '66)

(I) φ harmonic $\Leftrightarrow$ $\exists h \in C^\infty(M)$: $\Delta_g \varphi_i = h\varphi_i$; ~~$\Delta \varphi_i$~~
In this case $h = 2e(\varphi)$.

(II) φ isometric immersion, then:

$$\varphi \text{ minimal} \Leftrightarrow \Delta_g \Phi_i = m\,\Phi_i$$

Pf (II) follows from (I). $\varphi^* g_{round} = g \Rightarrow e(\varphi) = \frac{1}{2}\operatorname{tr}_g(\varphi^* g_{round})$
$= \frac{m}{2}$.

Pf of (I): $x\in M$, $\{e_i\}$ synch. frame.

$\tau(\Phi)(x) = \operatorname{tr}(\nabla d\Phi) = \sum_i \nabla^{\mathbb{R}^{n+1}}_{d\Phi(e_i)} d\Phi(e_i)(x)$
$\Delta^M \Phi(x)$.

$= \sum_i d\iota(\nabla^S_{d\varphi(e_i)} d\varphi(e_i))(x) - g_{S^n}(d\varphi(e_i), d\varphi(e_i))\Phi(x)$

$= d\iota(\tau(\varphi))(x) + 2e(\varphi)\Phi(x)$.

φ harmonic $\Leftrightarrow$ $\tau(\varphi) = 0$.

2 dim: $\Phi: M \to S^n \subset \mathbb{R}^{n+1}$ eigenmapping $\Leftrightarrow$ $\Delta_g \Phi_i = \lambda \Phi_i$ $\lambda\in\mathbb{R}$.

Goal: Determine eigenmappings for $(M,g) =$ Riem homo. space.