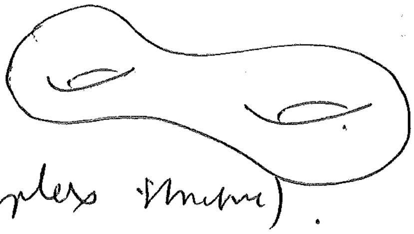
Colin Guillarmou - 2D conformal methods. 16/01/2017.

M surface genus g  oriented

g Riem metric ($\Rightarrow$ complex structure).

$[g] = \{ e^{\varphi} g ;\ \varphi \in C^\infty(M) \}$ conf. class of g ($\Leftrightarrow$ cx structure).

(Gawedzki) Def: a conformal field theory (CFT) are the following objects:

$Z : \mathrm{Met}(M) \to \mathbb{R}^+$ partition function of the CFT.

(I) $Z(e^{\varphi} g) = Z(g) \exp\left(\frac{c_Z}{96\pi} \int_M |d\varphi|_g^2 + 2k_g \varphi \, dv_g \right).$

k_g - scalar curvature of g

c_Z - central charge of the theory.

and $\forall\, \Phi \in \mathrm{Diff}(M)$. $Z(\Phi^* g) = Z(g)$.

(II) correlation functions: $x_1, \dots, x_N \in M$, $\underbrace{\alpha_1, \dots, \alpha_N}_{\text{weights}} \in \mathbb{R}$.

$Z(g ; \underbrace{\varphi_{\alpha_1}(x_1)}_{\text{primary fields}}, \dots, \varphi_{\alpha_N}(x_N)) \in \mathbb{R},$

Ideally: $Z(g) = \int_E e^{-S_g(\varphi)} \underbrace{D\varphi}_{\text{uniform formal measure}}$ | E infinite dim mfd. $E \cong$ maps $(M \to \mathbb{R})$.

$S_g : E \to \mathbb{R}$ action

$$Z(g; \mathcal{V}_{\alpha_1}(x_1), \dots, \mathcal{V}_{\alpha_N}(x_N)) = \int_{\mathcal{E}} e^{-\sum_{i=1}^N \alpha_i \varphi(x_i)} e^{-S_g(\varphi)} D\varphi.$$

$$Z(e^{\varphi} g ; \mathcal{V}_{\alpha_1}(x_1), \dots, \mathcal{V}_{\alpha_N}(x_N)) = Z(g; \mathcal{V}_{\alpha_1}(x_1), \dots, \mathcal{V}_{\alpha_N}(x_N)).$$

$$\times \exp\left(\frac{c_L}{96\pi} \int_M (|d\varphi|_g^2 + 2k_g\varphi)\, dv_g\right)$$

$$\times \exp\left(\sum_{i=1}^N \Delta_i(\alpha_1, \dots, \alpha_N)\, \varphi(x_i).\right).$$

$\in \mathbb{R}$, weights.

<u>Renormalized volume of 3 dim hyperbolic mfds.</u> $M = M^2$.

Gauss-Bonnet $\int_M k_g \cdot dv_g = 4\pi(2-2G)$.

Conformal change for k_g: if $\hat{g} = e^{w} g$.

$$k_{\hat{g}} = e^{-w}(\Delta_g w + k_g).$$

Th^m. $\forall g$ metrics on M, $\exists! \, w \in C^\infty(M)$ s.t. $\hat{g} = e^w g$ has scalar curv. $k_{\hat{g}} = -2$.

$(M, \hat{g})$ can be realized as $\Gamma \backslash \mathbb{H}^2$ $\mathbb{H}^2$ hyp. half plane,

$\Gamma \subset PSL_2(\mathbb{R})$. discrete subgroup.

<u>Pf</u>. use variational method:

Let $F_g : C^\infty(M) \to \mathbb{R}$,

$$\varphi \mapsto \int_M \left(\tfrac{1}{2} |d\varphi|^2 + k_g \varphi + 2e^{\varphi} \right) dv_g .$$

Minimize F_g, find φ_0, $\hat{g} = e^{\varphi_0} g$ has $k_{\hat g} = -2$.

φ_0 - minimi minimiser, critical pt.

<u>Remark</u>. if we fix vol.$(M, \cancel{g}\, e^{\varphi} g) = 1$. $\int_M e^{\varphi} dv_g = 1$,

you can get φ_0 by minimizing $\int (\tfrac{1}{2} |d\varphi|^2_g + k_g \varphi) dv_g$.

$\Rightarrow$ partition fcn. of a CFT is minimized at ~~a~~ the metric with scalar curvature $= -2$. So then

$$\int_M k_g \, dv_g = \int -2 \, dv_g \overset{GB}{=} 4\pi (2-2G).$$

Now let (X, g) be a 3-dim hyp. mfld non-cpct but conformally compact. i.e. $\exists\, \bar{X}$ mfld with bdry s.t. $\partial \bar{X} = M$ ~~compact~~, and with $\bar{X}$ compact.

$\exists\, \rho : C^\infty(\bar{X}) \to \mathbb{R}^+$ bdry defining function.

$\{\rho = 0\} = \partial \bar{X}$, $\rho > 0$ in X, $d\rho$ does not vanish on $\partial \bar{X}$

$$\mathring{\bar{X}} = X.$$

s.t. $\rho^2 g$ is a smooth metric on $\overline{X}$.

Rem. a short computation using $\mathrm{curv}(g) = -1$ near $\partial\overline{X}$:

$|d\rho|_{\rho^2 g} = -1$ at $\partial\overline{X}$. (Assume X oriented).

X can be realized as $\Gamma\backslash\mathbb{H}^3$, $\Gamma \subset \mathrm{Isom}_+(\mathbb{H}^3) = PSL_2(\mathbb{C})$.

$PSL_2(\mathbb{C})$ acts on $\mathbb{B}^3 = \{ m \in \mathbb{R}^3 \ |m| \le 1\}$ as conformal transform. Let $\Lambda(\Gamma) = \overline{\{\gamma(x) : \gamma \in \Gamma\}}^{\mathbb{B}^3}$ limit set of $\Gamma \subset S^2 = \partial\mathbb{H}^3$.

Let $x \in \mathbb{H}^3$; $\Omega(\Gamma) = S^2 \setminus \Lambda(\Gamma)$ open; Γ acts properly discontinuously.

Let

$\overline{X} = \Gamma\backslash(\mathbb{B}^3 \setminus \Lambda(\Gamma))$. proper & discontinuously.

$\partial\overline{X} = \Gamma\backslash\Omega(\Gamma) = \bigsqcup_{i=1}^N M_i$ ← connected Riemann surfaces. It has the complex conform. str. of $\Gamma\backslash\Omega(\Gamma)$.

M_i has a <u>projective structure</u>:

i.e. an atlas with charts $\varphi_j : M_i \to \mathbb{CP}^1$; $\varphi_k \circ \varphi_j^{-1}$ is in $PSL_2(\mathbb{C})$.

Get conformal classes on $M = \bigcup_{i=1}^N M_i = \partial\overline{X}$,

$[\underbrace{\rho^2 g|_{TM}}_{\bar{g}}]$ conformal boundary.

Lemma (Graham) $\forall h_0 \in [\rho^2 g|_{T\partial M}]$, $\exists!$ near $\partial\bar{X}$ bdy defining function $\hat{\rho} = e^{\omega}\rho$ s.t. $|d\hat{\rho}|_{\hat{\rho}^2 g} = 1$ near $\partial\bar{X}$ and $\hat{\rho}^2 g|_{T\partial M} = h_0$.

Consequence: if we take the flow of $\nabla^{\hat{\rho}^2 g}\hat{\rho}$; we get a diffeo: $\varphi : (0,\epsilon)_{\hat{\rho}} \times \partial\bar{X} \to \mathcal{U} \subset \bar{X}$.

$$\varphi^* g = \frac{d\hat{\rho}^2 + h(\hat{\rho})}{\hat{\rho}^2},$$ h is a family of metrics on $\partial\bar{X}$.

Lemma (Fefferman-Graham). $h(\hat{\rho}) = h_0 + \hat{\rho}^2 h_2 + \hat{\rho}^4 h_4$.

h_2; tensors on $\partial\bar{X}$. with $\mathrm{Tr}_{h_0}(h_2) = -\frac{1}{2}K_{h_0}$.

and $\delta_{h_0}(h_2) = \frac{1}{2} dK_{h_0}$.

$h_4 = \frac{1}{4} h_2^2$ (h_2^2 defined by consider h_2 as endo)

h_2 formally underdetermined, but trace is determined.

Colin Guillarmou Lecture 2. 19/01/2017.

Lemma (Graham) - Recall: (X,g) conformally cpct, hyperbolic 3-mfd, ρ bdry defining function, $\rho^2 g =: \bar{g} \in C^\infty(\bar{X})$ on cpctification $[\bar{g}|_{T\partial\bar{X}}] \ni \hat{h}_0$, $\exists!$ (near bdry) $\hat{\rho}$ b.d.f. s.t. $|d\hat{\rho}|_{\hat{\rho}^2 g} = 1$ near $\partial\bar{X}$ & $\hat{h}_0 = \hat{\rho}^2 g|_{T\partial\bar{X}}$ on.

Pf. Let $\hat{\rho} = e^w \rho$, $w \in C^\infty(\bar{X})$, and $|d\hat{\rho}|_{\hat{\rho}^2 g} = 1$.

$$\Leftrightarrow \quad 1 = \underbrace{\bar{g}(d\rho, d\rho)}_{\overset{\shortparallel}{1+\rho Q}\text{, Q smooth}} + \rho^2 |dw|^2_{\bar{g}} + 2\bar{g}(d\rho, dw)\rho.$$

$\Leftrightarrow$

+ condition $w_0 = w|_{\partial\bar{X}}$, then $e^{2w|_{\partial\bar{X}}} h_0 = \hat{h}_0$
$$\overset{\shortparallel}{e^{2w_0} h_0} = \hat{h}_0$$

$$\Leftrightarrow \quad 2(\nabla^{\bar{g}} \rho)(w) + Q + \rho |dw|^2_{\bar{g}} = 0.$$

Non-degenerate Hamilton-Jacobi eq^n.

Use grad-flow of $\hat{\rho}$ w.r.t. $\hat{\rho}^2 g$; get diffeomorphism

$$\Psi : [0,\varepsilon]_{\hat{\rho}} \times \partial\bar{X} \to U \subset \bar{X} \quad \text{s.t.}$$

$$\Psi^* g = \frac{d\hat{\rho}^2}{\hat{\rho}^2} + \frac{\hat{h}(\hat{\rho})}{\rho^2} \longleftarrow \text{metrics on } \partial\bar{X}.$$

$\hat{\rho}$ is called as geodesic boundary defining function for (X,g) associated to $\hat{h}_0$.

Now let ρ and $\hat{\rho}$ are two ~~the~~ geodesic b.d.f, let

$$\hat{\rho} = e^w \rho, \qquad \nabla^{\rho^2 g}_\rho =: \partial_\rho.$$

$$2\partial_\rho w = -\rho\left(|\partial_\rho w|^2 + |d_y w|^2_{h(\rho)}\right). \quad \text{with } w|_{\rho=0} = w_0.$$

y coordinates on $\partial\bar{X}$.

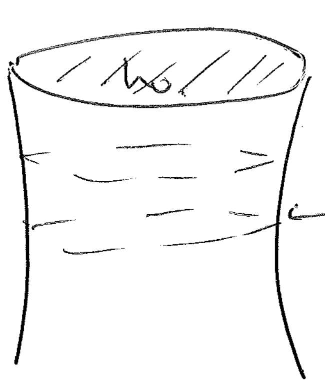
Taylor of w:
(at $\rho=0$).

$$w(\rho,y) = w_0(y) + \rho\, w_1(y) + \rho^2 w_2(y) + O(\rho^3).$$

$\Rightarrow$ get

$$4\rho w_2 + O(\rho^3) = -\rho\left(4\rho^2 w_2^2 + |d_y w_0|_{h_0} + 2\rho^2\langle dw_0, dw_2\rangle_{h_0} + O(\rho^3)\right)$$

$$\Rightarrow w_2 = -\frac{1}{4}|dw_0|^2_{h_0} \text{ and } w_1 = 0.$$



Can think of ρ as giving a certain foliation.

If ρ bdf associated to $h_0 \in [\rho^2 g|_{T\partial\bar{X}}]$, then

$$h(\rho) = h_0 + \rho^2 h_2 + \rho^4 h_4.$$

$$h_4 = \frac{h_2 \circ h_2}{4} \quad (h_2 \text{ identified with } \mathrm{End}(T\partial\bar{X})).$$

and
$$\begin{cases} \mathrm{Tr}_{h_0}(h_2) = -\frac{1}{2}K_{h_0}. \\ \delta_{h_0}(h_2) = \frac{1}{2}dK_{h_0}. \end{cases}$$
scalar curvature.

h_2 non-local in h_0 when solving Einstein eq. in X with bdy condition $[h_0]$.

Choose h_0 with $K_{h_0} = 2. \Rightarrow \mathrm{Tr}_{h_0}(h_2) = -1$ and $\delta_{h_0}(h_2) = 0$.

Teichmuller space of M h_2^{tt} is div free, trace free tensor (for h_0). is $\mathcal{T}(M) = \mathrm{Met}_{-1}(M) / \mathrm{Diffeo}_0(M)$

hyperbolic ; diffeo homotopic to identity.

$\dim_{\mathbb{R}} \mathcal{T}(M) = 6g-6$. $\mathbb{C}$-mfld , $T^*\mathcal{T}(M) = \{ k$ sym. 2-tens. s.t. $\mathrm{div}_{h_0}(k) = 0 = \mathrm{Tr}_{h_0}(k) \}$

Set of funnel ends / $\mathrm{Diffeo}_0(M) \simeq T^*\mathcal{T}(M)$.

<u>Renormalised volume</u>: the vol form of g near $\partial\bar{X}$ is

$$d\mathrm{vol}_g = \frac{d\rho \cdot d\mathrm{vol}_{h(\rho)}}{\rho^3} = \frac{d\rho}{\rho^3} d\mathrm{vol}_{h_0} \times N(\rho).$$

$$N(\rho) = \det(h_0^{-1} h(\rho)) = 1 + \rho^2 N_2 + O(\rho^4)$$

$$N_2 = \mathrm{Tr}_{h_0}(h_2) = -\tfrac{1}{2} k_{h_0}.$$

<u>Def</u>. (Renormalised vol) (X,g) hyp. 3-mfld, h_0 choice of metric at conformal bdy..

$$\mathrm{Vol}_R(X, g, h_0) = \underbrace{\mathrm{FP}_{z=0}}_{\text{Finite part, removing residue.}} \left[\int \rho^z \, d\mathrm{vol}_g \right].$$

<u>Ex.</u> $$= \mathrm{FP}_{\varepsilon \to 0} \underbrace{\int_{\rho > \varepsilon} d\mathrm{vol}_g}_{C_{-2}\varepsilon^{-2} + C_L \log(\varepsilon) + \mathrm{vol}_R(X,g,h_0) + O(\varepsilon)}.$$

Lemma $\mathrm{vol}_R(\bar X, g, e^{w_0} h_0) = \mathrm{vol}(\bar X, g, h_0) - \frac{1}{8}\int \left(\frac{|dw_0|^2_{h_0}}{2} + k_{h_0} w_0\right) dv_{h_0}$

"Conformal anomaly of a CFT."

Hint of pf: $\mathrm{FP}_{z=0} \int \hat\rho^{z}\, d\mathrm{vol}_g = \mathrm{FP}_{z=0} \int \rho^z \underbrace{e^{zw}}_{1+zw+O(z^2)} v(\rho)\, d\rho\, d\mathrm{vol}_{h_0}$

$= \mathrm{vol}_R(X, g, h_0) + \underbrace{\mathrm{Res}_{z=0} \int \rho^{z-3} N(\rho)\, w\, dv_{h_0}\, d\rho}$

$\mathrm{Res}_{z=0} \int \rho^z (w_2 + w_0 N_2) \frac{d\rho}{\rho}\, d\mathrm{vol}_{h_0} = \int_{\partial\bar X} (w_2 + w_0 N_2)\, d\mathrm{vol}_{h_0}$.

If $Z(g)$ is partition functn of CFT, define the tensor
$H_g \in C^\infty(M, \otimes^2_s T^*M)$, st

$$\langle d(\log Z(g)), h \rangle = \int_M \langle H_g, h\rangle_g\, dv_g.$$

h: sym 2-form.

$$Z(e^w g) = Z_g\, e^{\frac{c}{96\pi}\int_M (|dw|^2_g + 2 k_g w)\, dv_g} \quad (1)$$

$Z(\Phi^* g) = Z(g)$ $\forall \Phi \in \mathrm{Diff}(M)$. (2)

(2) $\Rightarrow$ $\delta_g(H_g) = 0$ since $\langle H_g, \mathcal{L}_X g\rangle = 0$ $\forall X$ v. fields.

(1) $\Rightarrow$ $\mathrm{Tr}_g(H_g) = \frac{2c}{96\pi} k_g$.

Call H_g stress-energy tensor.

If g_t is a family of hyp metrics on M,

$\partial_t|_{t=0} Z(g_t) = H_{g_0} \Rightarrow H^{tr}_{g_0}$ is div-free.

Tr-free: $H^{tf}_{g_0} \in T^*_{g_0}\mathcal{T}(M)$.

Ex $\mathrm{vol}_R(X, g, \Phi^* h_0) = \mathrm{Vol}_R(X, g, h_0)$.

$\forall \Phi$ homotopic to identity!

Define $\mathrm{vol}_R(X,g) = \mathrm{Vol}_R(X, g, h_0)$ ← hyperbolic in conformal bdry.

Th^m If g_t is a family of conformally compact hyp 3-mfld,

$\partial_t|_{t=0} \mathrm{vol}_R(X, g_t) = -\frac{1}{4}\int_{\partial\bar{X}} \langle h_2^{tf}, \dot{h}_0\rangle_{h_0} \, dvol,$

$\dot{h}_0 = \partial_t|_{t=0} h_0^t$. h_0^t hyp metric at $\partial\bar{X}$ for g_t.

Pf. Use Schläffli formula: $\partial_t \mathrm{vol}_R(X, g_t) = \frac{1}{n}\int_{\partial\bar{X}} (\dot{H} + \frac{1}{2}\langle \dot{g}, \mathrm{II}\rangle_g)\, dv_g$

t deriv. of mean curv. ↑ 2nd ff. at $\partial\bar{X}$.

If g_t is Einstein on cpct mfld with bdy (dim $n+1$).

* Can view Vol_R as a fn on Teichmüller space over boundary;

Th^m: $\bar{\partial}\partial \mathrm{vol}_R(X) = c\, \omega_{WP}$ ← Weil-Petersson. Symplectic form.

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Z partition function of 2D CFT $\quad \text{Metrics}(M) \to \mathbb{R}$.

- Conformal anomalies, diffeo invariance.

$\to$ Z minimised at $g = g_{hyp}$ hyperbolic metric.

$dZ(g_{hyp})$ in the direction of Teichmüller space $T(M) = \mathrm{Met}_{-1}(M)/\mathrm{Diff}_0(M)$

$\quad \| $

$\det g_{hyp}$, $H^{tt}_{g_{hyp}}$ is traceless & div. free, $\in T^*_{g_{hyp}} T(M)$.

"Stress-Energy" tensor

Thm. $T^* T(M) \xrightarrow{\text{iso}} P(M)$ = set of projective structures on M.

From $Z \longrightarrow$ a projective structure. section of $P(M) \to T(M)$ (affine bundle).

(see survey by David Dumas on Proj. structures)

(II). LIOUVILLE QFT.

(M,g). Riem. surface, Liouville action:

$$S_L(g,\varphi) := \frac{1}{4\pi}\int \left(|d\varphi|^2_g + Q k_g \varphi + 4\pi\mu e^{\gamma\varphi}\right) dv_g .$$

$S_L(g,\cdot)$ is minimised at φ_0 s.t. $g' = e^{\gamma\varphi_0} g$ has constant scalar curvature if $Q = \frac{2}{\gamma}$, $k_{g'} = -2\pi\mu\gamma^2$.

Liouville QFT:

$$Z(g) := \int_{\substack{E \\ \shortparallel \\ \{\text{maps } M\to\mathbb{R}\}}} e^{-S_L(g,\varphi)} D\varphi \qquad \text{Formal integral.}$$

More generally: $Z(g,F) := \int F(\varphi)\, e^{-S_L(g,\varphi)} D\varphi$, $F: E \to \mathbb{R}$,
in particular, with $F(\varphi) = \prod_{i=1}^{N} e^{\alpha_i \varphi(x_i)}$ primary fields.

$\boxed{\text{⚠ Rem: in physics, } Q \text{ should be } 2/\gamma + \gamma/2.}$

1) Gaussian Free Field: (cf. survey paper by Scott Sheffield).

$(\varphi_i)_{i\in\mathbb{N}}$ o.n. basis of eigenfunctions of Δ_g, Δ_g

$$\Delta_g \varphi_i = \lambda_i \varphi_i, \quad \lambda_0 = 0,\ \varphi_0 = \frac{1}{\sqrt{\operatorname{vol}_g(M)}}.$$

$(\Omega, \mathcal{F}, \mathbb{P})$ probability space,

$(\alpha_i)_{i\in\mathbb{N}}$ family of Gaussian random variables which identically distributed independent $\mathbb{E}(\alpha_i)=0$, $\mathbb{E}(\alpha_i,\alpha_j)=\delta_{ij}$.

Def GFF: $$X_g(x) = \sum_{i=1}^{\infty} \alpha_i(\omega) \frac{\varphi_i(x)}{\sqrt{\lambda_i}}.$$
$\qquad\qquad\qquad\qquad\Omega$

Rem: $X_g \in H^{-s}(M)$ $\forall s>0$ a.s.

$$\mathbb{E}\left(\|(1+\Delta_g)^{-s/2} X_g\|_{L^2}\right) = \mathbb{E}\left(\sum_{i>0} \frac{\alpha_i}{(1+\lambda_i)^{s/2}\lambda_i^{1/2}}\right) = \sum_{i>0} \text{——} < \infty$$
by Weyl law.

Covariance of X_g : $\varphi, \varphi' \in C^\infty(M)$.

$$\mathbb{E}\left(\langle X_g, \varphi \rangle \langle X_g, \varphi' \rangle \right) = \dots = 2\pi \langle G_g, \varphi \otimes \varphi' \rangle_{M \times M}.$$

where $\quad G_g(x,x') = \sum_{i \geq 1} \dfrac{\varphi_i(x)\varphi_i(x')}{\lambda_i}$

kernel for op R_g $\quad \Delta_g R_g = Id - \Pi_0 \leftarrow$ project on $N(\Delta_g)$.

<u>Rem</u>. $G_g(x,x') = \dfrac{-1}{2\pi} \underbrace{\log(d_g(x,x'))}_{\text{Riem dist}} + \underset{\substack{\uparrow \\ \text{cts} \\ \text{"mass of } g\text{"}}}{m_g(x)} + o(1)$. $\quad x \mapsto x'$.

Let $H_0^{-s}(M) = \{ \varphi \in H^{-s}(M) : \langle \varphi, 1 \rangle_g = 0 \}$.

<u>Lemma</u>. $\exists$ probability P on $H_0^{-s}(M)$ s.t. the law of random var X_g is P and $\forall \varphi \in H^s(M)$, $\langle X_g, \varphi \rangle$ is a random (gaussian) var on Ω with zero mean and covariance

$2\pi \langle G_g, \varphi \otimes \varphi \rangle$.

$$\mathcal{U} \subset H_0^{-s}(M), \quad \mathbb{P}(X_g \in \mathcal{U}) = P(\mathcal{U}).$$

P represents the formal measure $F \mapsto \int F(\varphi) e^{-\frac{1}{4\pi}\int_M |d\varphi|^2} D\varphi$

$$\times \sqrt{\det\left(\tfrac{1}{2\pi}\Delta_g\right)}.$$

Approximate on $1 \leq j \leq N$:

$$\int F(\varphi) e^{-\frac{1}{4\pi}\|d\varphi\|_{L^2}^2} D\varphi = \int_{\mathbb{R}^N} F\Big(\sum_{1\leq j \leq N} \alpha_j \varphi_j\Big) \prod_{i=1}^N e^{-\frac{1}{4\pi}\alpha_i^2 \lambda_i} d\alpha_i.$$

$\delta > 0$ fixed, To get a measure on $H^{-s}(M)$, take

$$\varphi : \begin{array}{l} H_0^{-s}(M) \times \mathbb{R} \xrightarrow{\text{iso}} H^{-s}(M) \\ (X, c) \longmapsto X + c \end{array}$$

$H^{-s}(M)$ ← indep of g as a vec. space.

$H_0^{-s}(M)$ ← depends on g!

Def. Let $P' = \varphi_*(P \otimes dc)$ ← Lebesgue.

__Lemma:__ P' on $H^{-s}(M)$ is conformally invariant:

if $\hat{g} = e^{\omega} g$; $\int_{\mathbb{R}} \mathbb{E}(F(X_{\hat{g}} + c))\, dc = \int_{\mathbb{R}} \mathbb{E}(F(X + c))\, dc$.

__Pf__. use relation b/w $G_{\hat{g}}$ and G_g.

P' represents: $\sqrt{\det(\frac{1}{2\pi}\Delta_g)} \cdot \underbrace{e^{-\|d\varphi\|_{L^2}^2}}_{\text{weird, } \varphi \in H^{-s}, \text{ but okay } \ldots \text{ by some magic.}} D\varphi$.

is a measure on $H^{-s}(M)$.

To define LQFT: partition function

$$\Pi_{\mu, \gamma}(g) = \left(\frac{\det \Delta_g}{\operatorname{vol}_g M}\right)^{-\frac{1}{2}} \int_{\mathbb{R}} \mathbb{E}\left(e^{\underbrace{-\frac{Q}{4\pi}\int_M K_g (X_g + c)}_{\text{makes sense since } K_g \in C^\infty}\; - \mu \overbrace{e^{\gamma(c + X_g)}\, dv_g}^{\text{does not make sense.}}}\right).$$

with $Q = \frac{2}{\gamma} + \frac{\gamma}{2}$.

go through renormalisation. since $e^{\gamma(c + X_g)}$ problem. here.

↑ distribution.

2) Gaussian multiplicative chaos. (GMC).

(Kahane 85, Rhodes-Vargas 2008, Duplantier-Sheffield '09).

Let $X_{g,\varepsilon}(x) := \int X_g \cdot d\mu_{x,\varepsilon}$ — uniform measure on the geodesic circle of radius $\varepsilon > 0$ · centre $x \in M$ for g.

cts in x a.s.

<u>Prop</u>. If $\gamma \in (0,2)$, then the measure

$$g^{\gamma}_{g,\varepsilon}(x) = \varepsilon^{\gamma^2/2} e^{\gamma X_{g,\varepsilon}(x)} \, dv_g(x).$$

Converges to a measure (in prob & weak sense).

g^{γ}_g called GMC.

Define part. funct. using $M e^{\gamma(c + X_g)}$ replaced by $M e^{\gamma c} g^{\gamma}_g$:

<u>Thm</u>. $\Pi_{\mu,\gamma}$ is the partition function of a CFT with central charge $1 + 6Q^2$ if $Q = \frac{\gamma}{2} + \frac{2}{\gamma}$

& can define correlation functions in the same way.