DG (III) . Ahmed Afzani 13/10/2017.

Notation: (à la Warner).

Sps M^n, N^m smooth mflds.

- $x \in M$ T_xM tangent space at x.

  T_x^*M cotangent.

  TM / T^*M. bundles.

- $\varphi = (x^1, x^2, x^n) : U \subset M \to \varphi(U) \subset \mathbb{R}^n$ · chart.

  (φ^{-1} local param.).

  $\left\{ \frac{\partial}{\partial x^i}\Big|_p \right\}_{i=1}^n$ resp. $\{ dx^i|_p \}_{i=1}^n$ assoc. basis of T_pM/T_p^*M

  (Also write $(x^1, \ldots, x^n)$ even in M).

- $f : M \to N$ C^∞, $d_xf : T_xM \to T_{f(x)}N$. diff at x.

  $\delta_x f : T^*_{f(x)}N \to T^*_x M$ co diff.

*If V is a v.s., V^* dual, then bil.

$$\underbrace{(V^* \otimes \cdots \otimes V^*)}_{k} \times (V \otimes \cdots \otimes V) \to \mathbb{R}.$$

$$(\nu_1 \otimes \cdots \otimes \nu_k, v_1 \otimes \cdots \otimes v_k) \to \nu_1(v_1) \cdots \nu_k(v_k).$$

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* No metrics at this stage, this is done pointwise!! as v.s. duals.

Motivation. [Warner, Bishop & Goldberg "Tensor analysis on mflds"

- Bochner method: have "Laplacian" on Riem mfld.

  (M,g). $\Delta_g = C^\infty(M) \to C^\infty(M)$.

for one-forms $\omega \in C^\infty(M \to T^*M)$, have another Laplacian $\bar{\Delta}$ where null space $\{\omega : \Delta\omega = 0\}$ characterises first cohomology.

$\rightsquigarrow$ $\Delta_g\left(\frac{1}{2}|\omega|^2\right) = |\nabla\omega|^2 + Ric(\omega,\omega)$.

(Hessian in ↑).

$\longrightarrow$ Δ. integrate: $0 = \int_M |\nabla\omega|^2 + \int_M Ric(\omega,\omega)$.

$Ric > 0 \Rightarrow \omega = 0$.

(recall ↑ cpt).

- Scalar curvature: $\rightsquigarrow$ Spinors.

  "Spinor model belts fixed on one end".

- Symmetries: $X(p) = \sum_{i=1}^{n} X^i \partial_i|_p \in T_pM \iff \tilde{X}^i = \sum_{j=1}^{n} \frac{\partial \tilde{x}^i}{\partial x^j} X^j$.

  $= \sum_{i=1}^{n} \tilde{X}^i \tilde{\partial}_i|_p$. where $(\tilde{x}^1, \dots, \tilde{x}^n)$ is another coord. system.

$\left(\frac{\partial \tilde{x}^i}{\partial x^j}\right) \in GL(n, \mathbb{R})$.

$\rightarrow$ crucial to go through G-bundles.

Integrable systems

Problem: Find a function $f \in C^\infty(U, \mathbb{R})$ $U \subset \mathbb{R}^n$.

s.t.
$$\begin{cases} \sum_{j=1}^{n} X_1^j(x)\, \partial_j f(x) = 0. \\ X_{j=1}^{n} X_2^j(x)\, \partial_j f(x) = 0. \end{cases}$$

i.e., f solves a system of eq^{n}s.

Examples: • $\frac{\partial}{\partial x} f = 0$. $n = 3$ $\rightsquigarrow$ $f(x,y,z) = g(y,z)$.

• $\underbrace{\frac{\partial}{\partial x} f + \frac{\partial}{\partial y} f}_{\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right) f} = 0.$ $(n=3)$ method of characteristics.

$\rightsquigarrow$ look for sol^n to $\dot{\gamma}(t) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\gamma(0) = q$m.

$\rightarrow$ $\Rightarrow \frac{d}{dt}(f \circ \gamma) = 0.$

$x - y = $ const. So gen sol^n $f(x,y,z) = g(x-y, z)$.

More generally: look for $f \in C^\infty(U, \mathbb{R})$ $U \subset M$.

s.t.
$$\begin{cases} \partial_{x_1} f = 0 \\ \vdots \\ \partial_{x_d} f = 0 \end{cases}$$

for $\mathbb{R}$-linearly indep v. fields $\{x_i\}_{i=1}^{d}$.

Def^n A d-distribution is tangent subbundle of TM

i.e., a collection $D = \bigcup_{x \in M} D_x \subset TM$.

$\forall x_0 \in M$ $\exists$ a nbh $V \ni x_0$ & d vec fields $\{X_i\}_{i=1}^{d}$ s.t. $D_x = \operatorname{span}_{\mathbb{R}} \{X_i(x)\}_{i=1}^{d}$.

(*) Note. pde before local, distributions are local.

Def^n. An integral submfld of D is a submfld $S \overset{i}{\hookrightarrow} M$ s.t. $\forall s \in S$ $d_s i(T_s S) = D_{i(s)}$.

If for all $p \in M$, $\exists$ an integral submfld S of D passing through p (i.e., $\exists s \in S$ s.t. $i(s) = p$), then D is completely integrable.

Ex. consider $\mathbb{R}^3$,
$$\begin{cases} X_1(x,y,z) = \partial_x + z\,\partial_y \\ X_2(x,y,z) = \partial_z \end{cases}$$

Is this system completely integrable

~~Df~~ Lemma If D is completely integrable, then $\forall$ v.f. $X, Y : n \to D$, $[X,Y](x) \in D_x$.

Pf. Fix $p\in M$, S a integrable submfld. ~~Define~~ passing through p.

Defn $\underline{X}, \underline{Y} : S \to TS$. s.t. $\begin{cases} di(\underline{X}) = X\circ i & \text{and} \\ di(\underline{Y}) = X\circ i \end{cases}$

$$\mathcal{D}_{i(s)} \ni di\,[X,Y](s) = [X,Y](i(s)). \quad \text{In particular for } i(s)=p.$$

Ex. $[X_1, X_2] = [\partial_x + z\partial_y, \partial_z] = -\underset{0}{\cancel{\partial_z(1)}}\cdot\partial_x \cancel{=0} - \partial_z(z)\partial_y.$

∂_z $\quad = -\partial_y$

$\notin \text{span}\{X_1, X_2\}$.

Defn If a distr. D is s.t. $\forall$ v.f. $X, Y : M \to D$, $[X,Y]\in D$, ~~then~~ D is involutive.

Thm (Frobenius) If a dist. D is involutive, it is completely integrable. viz $\forall p\in M$, $\exists$ a ~~completely~~ coord. sys. $\tilde{\varphi} = (z^1, \dots, z^n) : \tilde{V} \to \tilde{\varphi}(\tilde{V})$ s.t. the 'slices' $\{z^{d+1} = \alpha_1, \dots, z^n = \alpha_{n-d}\}$ are all integral submflds for fixed $\alpha_1, \dots, \alpha_{n-d} \in \mathbb{R}$.

Pf (by induction) Fix $p\in M$.

$d=1$: solve ODE system $\begin{cases} \dot{\gamma}(t) = X(\gamma(t)). \\ \gamma(0) = p. \end{cases}$

$\rightsquigarrow$ flow φ_t

fix a coord system $\begin{cases} \varphi = (x^1, \dots, x^n) : V \to \varphi(V) \quad \text{s.t.} \\ \cdot\ \varphi(p) = 0, \quad d_p \varphi(X(p)) = \frac{\partial}{\partial x^1}\Big|_0 . \end{cases}$

Now define for small $\delta > 0$.

$$\begin{cases} F : (-\delta, \delta)^n \to M . \\ \quad (t^1, \dots, t^n) \mapsto \varphi_t (\varphi^{-1}(0, t^2, \dots t^n)) . \end{cases}$$

$$\rightarrow \frac{\partial}{\partial t^1}\Big|_{(t^1, \dots, t^n)} = X(F(t^1, \dots, t^n)) .$$

Ex: check F is a diffeo for small δ.

Sps ~~A~~ true for $d-1$: fix a local basis $\{X_i\}_{i=1}^d$ for D, let $\varphi : U \to \varphi(U)$ be a coord sys. on M s.t.

$$\begin{cases} \cdot\ d\varphi(X_1) = \frac{\partial}{\partial x^1} \quad (\text{poss. from } d = 1 \text{ case}) . \\ \cdot\ U \subset \text{dom of } X_i \ \forall i \\ \cdot\ \varphi(p) = 0 . \end{cases}$$

Define: $Y_1 = X_1$, $Y_i = X_i - \partial_{X_i}(x^1)\, X_1 \quad (i \geq 2)$.

$\rightarrow \{Y_i\}_{i=1}^d$ local basis for D.

Also, $\partial_{Y_i} x^1 = \partial_{X_i} x^1 - \partial_{x_i} x^1 \cdot \partial_{x_1} x^1 = 0$.

$\Rightarrow \partial_{[Y_i, Y_j]} (x^1) = 0 \quad (i \geq 2)$, $\partial_{[Y_1, Y_i]} x^1 = 0$.

$\rightarrow [Y_i, Y_j] \in \text{span} \{Y_2, \dots, Y_d\}_{i,j \geq 2}$,

$[Y_1, Y_i] \in$ span $\{Y_2, \dots, Y_d\}$

Now set $N = \{x^1 = 0\} \overset{j}{\hookrightarrow} M$, obtain v.f. ($p \in N$).

Obtain v.f. $\{\underline{Y}_i : N \to TN\}_{i=2}^{d}$ s.t. $dj(\underline{Y}_i) = Y_i \circ j$, $i \geq 2$

Claim. span $\{\underline{Y}_2, \dots, \underline{Y}_d\}$ is an involutive distr. on N of dim $(d-1)$.

'<u>pf</u>' $dj([\underline{Y}_i, \underline{Y}_j]) = [Y_i, Y_j] \circ j = \sum_{k=2}^{d} \alpha_{ij}^{k} \circ j (Y_k \circ j)$.

Induction hyp $\Rightarrow$ $\exists$ coordinates $\psi = (y^2, \dots, y^n)$ in a nbhd of p s.t. $\psi(p) = 0$ & for $\alpha_1, \dots, \alpha_{n-d} \in \mathbb{R}$ $\{y^{d+1} = \alpha_1, \dots, y^n = \alpha_{n-d}\}$ integral.

Now define $F(t^1, \dots, t^n) = \varphi^{-1}(t^1, (pr_2 \circ \varphi \circ \psi^{-1})(t^2, \dots, t^n))$

$pr_2 : \mathbb{R}^n \xrightarrow{=} \mathbb{R} \times \mathbb{R}^{n-1} \to \mathbb{R}^{n-1}$ projection.

F local param of M in nbhd of $p \in M$.

$\Rightarrow$ $\tilde{\psi}(m) = F^{-1}(m) = (x^1(m), \psi(\varphi^{-1}(0, pr_2 \varphi(m))))$.

Need to check: $\partial_{Y_i}(z^j) = 0$ $\forall j \in \{d+1, \dots, n-d\}$, $i \in \{1, \dots, d\}$.

$i = 1$ $\partial_{Y_1} z^j = \partial_{z^1} z^j = 0$.

$i > 1$. ~~[illegible]~~ On $z^1 = 0$, $\tilde{\psi}(m) = (0, \psi(\varphi^{-1}(0, pr_2 \varphi(m))))$

but $x \in \{z_1 = 0\} \Rightarrow \tilde{\varphi}(x) = (0, \varphi(x))$.

$$\Rightarrow \partial_{y_i} z^j = \partial_{y_i} y^j = 0 .$$

why is $\{z_1 = 0\}$ enough?: $\frac{\partial}{\partial z^1}(\partial_{y_i} z^j) = \partial_{[Y_1, Y_j]} z^j = \partial_{\left(\sum_{k=2}^{d} \beta_j^k z_k\right)} z^j$

$$= \sum_{k=2}^{d} \beta_j^k (\partial_{y_k} z^j) .$$

uniqueness of sol^n to ODE $\Rightarrow$ $\blacksquare$.

Saskia Roos - Dirac Eigenvalues under codim 1 ~~c~~ collapse. 19/10/2017.

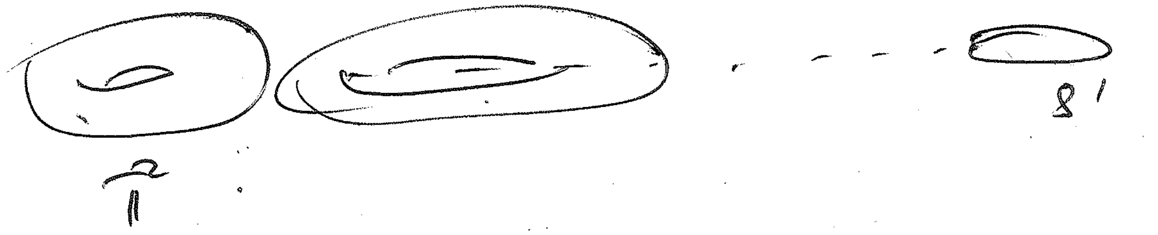
Def $\mathcal{M}(n,d) = \{(M,g) : \dim M = n, \ |\sec| \leq 1, \ \operatorname{diam} \leq d\}$

Thm $\mathcal{M}(n,d) \supset (M_i, g_i)$ has subsequence ~~precompact in GH topo~~ converging to GH metric space (N, g_N).

Note: If $\dim N = n$, then, $N \in C^\infty$ with $C^{1,\alpha}$ metric tensor.

What happens if dimension collapses?

Ex. T^2



Berger '62: small fibres of Hopf fib. $S^1 \to S^3 \to S^2$ by ε, $(S^3, g_\varepsilon) \xrightarrow{\varepsilon \to 0} S^2$.

$S^3 \simeq SU(2) \circlearrowleft \mathbb{Z}_4$ Riemannian action. $(S^3/\mathbb{Z}_4, g_\varepsilon)$

action $\begin{pmatrix} e^{i2\pi k/4} & \\ & e^{i2\pi k/4} \end{pmatrix}$.

$\downarrow \varepsilon \to 0$

orbifold!

Thm (Fukaya '90): If limit (N,h) has codim 1 $\Rightarrow$ (N,h) Riem orbifolds

Thm (Naber-Tian 2011). For any limit space (N,h) there is a closed set S s.t. $N \setminus S$ Riem orb.

$$\dim_{Haus}(S) \leq \min\{n-5, \dim(N)-3\}.$$

Thm (R '17). (M_i, g_i) in $\mathcal{M}(n,d)$ converging to (N,h), then TFAE:

① $\dim N \geq n-1$.

② $\forall r > 0$ $\exists C$ s.t. $C \leq \dfrac{\mathrm{vol}(B_r^{M_i}(x))}{\mathrm{inj}(M_i,x)}$ $\forall x$ $\forall i$.

③ $\exists r > 0$ $\exists C$ s.t. ______ ______.

Corollary. $\mathcal{M}(n,d,C) = \{(M,g) \in \mathcal{M}(n,d) : C \leq \mathrm{vol}(M)/\mathrm{inj}(M)\}$.

Suppose (M_i, g_i) collapses to (N,h), then

- $\dim(N) = n-1$, N is Riem orb., h is $C^{1,\alpha}$.
- $\mathrm{vol}(N) \geq v(n,d,C)$.
- $\|\sec(N)\|_{L^\infty} \leq k(n,d,C)$.

Intuition of this condition: inj is like 1 d, vol is n-d. ⇝ n-1 d.

But, can't do this for 2-d collapse by simply taking ~~inj^2~~ $\mathrm{inj}(M)^2$, because you could send speed of collapse in two diff. directions at diff. speed.

Thm (Cheeger-Gromov-Fukaya '92) adapted.

$(M,g) \in \mathcal{M}(n,d)$ and (N,h) an $(n-1)$-dim Riem orbifold, $\exists\, \varepsilon(n,d)$ s.t. if $d_{GH}(M,N) < \varepsilon$ then $\exists\, f : M \to N$ S^1-bundle with structure group $S^1 \rtimes \{\pm 1\}$ and $\exists$ new metric $\tilde{g}$ s.t. $\|g - \tilde{g}\|_{C^{1,5}} \leq c(n)\, d_{GH}(M,N)$ and $(M,\tilde{g}) \xrightarrow{f} (N,\tilde{h})$ Riem submersion.

Remark. M orientable & N not orientable.

$\to$ orientation cover $\hat{f} : \hat{M} \to \hat{N}$ S^1-principle bundle.

$f : M \to N$ S^1 principal, M spin.

Def. ~~Spin bundle~~ Spin structure on M proj if S^1-action lifts $P_{spin}(M)$. Non-proj if not.

Prop. M has proj spin structure $\Longleftrightarrow$ N is spin.

M has no proj spin $\Longrightarrow$ N is spin^c.

$\mathrm{Spin}^c(n) = \mathrm{Spin}(n) \times_{\mathbb{Z}_2} S^1$

Rem. there is a notion of spin structure for orbifolds N.

Note: $f: M \to N$ S^1 principal bundle, f Riem. submersion.

- k - killing field induced by S^1 action.
- $d = |k|$.
- F curvature of S^1-bundle.

$S^1 \circlearrowright P_{spin}(M)$. $\rightsquigarrow$ $\mathcal{L}_k : L^2(\Sigma M) \to L^2(\Sigma M)$.

$\mathcal{L}_k$ has eigenvalues ik, $\begin{cases} k \in \mathbb{Z} & \text{proj.}, \\ k \in (\mathbb{Z}+\frac{1}{2}) & \text{non-proj.} \end{cases}$

$\Rightarrow$ $L^2(\Sigma M) = \bigoplus_k V_k$. eigenspaces of $\mathcal{L}_k$.

$D(V_k) \subset V_k$ since D commutes with $\mathcal{L}_k$

$\longrightarrow$ number of eigenvalues of D as $\{\lambda_{j,k}\}_{j \in \mathbb{Z}}$. e.v. of $D|_{V_k}$.

Dirac op diagonal w.r.t. V_k.

"Proj. case": $\dim M = n+1 \Rightarrow \dim_{\mathbb{C}}(\Sigma M) = 2^{\lfloor \frac{n+1}{2} \rfloor}$

$\dim N = n$ even $\Rightarrow$ $\Sigma M \simeq f^*(\Sigma N)$.

odd $\Rightarrow$ $\Sigma M \simeq f^*(\Sigma N^+) \oplus f^*(\Sigma N^-)$

⊗ This is a way of spinors you can take. Same types: $\Sigma N \simeq \Sigma N^{\pm}$ and pull back to M, which is <u>even</u> dimensional.

<u>Prop</u>. (Ammann '98).

$$\forall k\ ;\ \exists\ \text{~~isom~~}.\quad Q: \begin{cases} L^2(\Sigma N \otimes L^{-k}) \to V_k & n \text{ even}. \\ L^2((\Sigma N^+ \oplus \Sigma N^-)\otimes L^{-k}) \to V_k & n \text{ odd} \end{cases}$$

L^{-k} line bundle, $L = M \times_{S^1} \mathbb{C}$.

$X \in \Gamma(TN)$, $\tilde{X}$ has lift.

$$\gamma(\tilde{X})\, Q_k(\varphi) = \begin{cases} Q_k(\gamma(X)\varphi), \\ Q_k(\gamma(X)\varphi^+ \oplus -\gamma(X)\varphi^-). \end{cases}$$

V unit vertical $\quad i\gamma(V)\, Q_k(\varphi) = \begin{cases} Q_k(\bar{\varphi}). \\ Q_k(\varphi^- \oplus \varphi^+). \end{cases}$

<u>Thm</u>. (<u>R '17</u>).

$(M_a, g_a)_a \in \mathcal{M}(n, d, C)$, ~~ca~~ $\to (N, h)$ s.t. Spin structures project. $\forall \varepsilon > 0\ \exists\ a_0$ s.t.

$\forall\, a > a_0$.

$$|\lambda_{j,k}(a)| \geq \frac{|k|}{\|l_a\|_\infty} - \frac{1}{2}\left[\frac{n}{2}\right]^{\frac{1}{2}} \|l_a F_a\|_\infty - \varepsilon.$$

$\|l_a\|_\infty \leftarrow$ ~~[illegible]~~ length of fibres.

Since $\|l_a\|_\infty \to 0$

$\forall k \neq 0 \Rightarrow |\lambda_{j,k}(a)| \to \infty$.

$\lambda_{j,0}(a)$ converge to the eigenvalues of

n even: $D^N\varphi + \frac{1}{4}\gamma(\tilde{F})\bar{\varphi}$. $\tilde{F}$ 2-form on N

n odd: $\begin{pmatrix} D^N & \frac{i}{4}\gamma(\tilde{F}) \\ \frac{i}{4}\gamma(\tilde{F}) & -D^N \end{pmatrix}$.

"$\tilde{F} = \lim_{a\to\infty} l_a F_a$".

"**<u>Pf</u>**".

$$D_a = \frac{1}{l_a}\gamma\left(\frac{u_a}{l_a}\right)\mathcal{L}_{u_a} + D^H - \frac{1}{4}\gamma\left(\frac{k_a}{l_a}\right)\gamma(l_a F_a).$$

$$D^H|_{V_k} = Q_k \circ D_{\Sigma N\otimes L_k} \circ Q_k^{-1}.$$