Kato for Parabolic systems. 27/05/2016.

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$\mathbb{R}^{n+1} \ni (x,t)$ with Lebesgue measure $dx\,dt$.
$x \in \mathbb{R}^n$, $t \in \mathbb{R}$.

$H = \partial_t - \operatorname{div}_x A(x,t)\nabla_x$.

$\sqrt{H}$?

$A \in L^\infty(\mathbb{R}^{n+1}, M_n(\mathbb{C}))$, $\operatorname{Re}\langle A(x,t)\xi,\xi\rangle \geq \lambda|\xi|^2$. $\xi \in \mathbb{C}^{n}$ a.e. (x,t).
$\lambda > 0$.

Th^m 1) H can be defined as a maximal accretive operator on $L^2(\mathbb{R}^{n+1})$ from a form with domain

$$V = \{ f \in L^2(\mathbb{R}^{n+1}) : \nabla_x f \in L^2,\ D_t^{\frac{1}{2}} f \in L^2 \}.$$

$D_t^{\frac12} \overset{\text{Fourier}}{\rightsquigarrow} |\tau|^{\frac12}$.

2) $\mathcal{D}(\sqrt{H}) = V$ with $\|\sqrt{H} f\|_2 \simeq \|\nabla_x f\|_2 + \|D_t^{\frac12} f\|_2$.

Comments: i) Elliptic version $L = -\operatorname{div}_x A(x)\nabla_x$.
'02 H.L.Mc.T. & A.

$\|\sqrt{L} f\|_2 \sim \|\nabla_x f\|_2$, $f \in H^1(\mathbb{R}^n)$.

- $T(b)$ argument,
- $A = A^*$, easy. $(\Rightarrow L = L^*)$.

2) on the statement.

- $\mathrm{Re}\,(Hf, f) = \mathrm{Re}\left(\iint \partial_t f \cdot \bar{f} + A\nabla_x f \cdot \overline{\nabla_x f}\right)$.

$$= \mathrm{Re}\left(\iint A\nabla_x f \cdot \overline{\nabla_x f}\right).$$

$$\geq k\|\nabla_x f\|_2^2 \geq 0.$$

- Invertibility of $1+H$?
- $A = A^* \not\Rightarrow H = H^*$, $\quad H^* = -\partial_t - \mathrm{div}_x A^* \nabla_x$.

* History

Implicit in K. Nyström (Adv. '16). Th$^{=}$ holds when $A(x,t) = A(x)$ $\rightarrow$ via square function estimates.

- Pf adopted elliptic pf.
- Strongly uses that $A(x,t) = A(x)$.

Definition of H

$D_t^{\frac{1}{2}} \xrightarrow{\text{Fourier}} |\tau|^{\frac{1}{2}}$, $\quad H_t \rightsquigarrow i\,\mathrm{sgn}\,\tau$. Hilbert transform.
(acts on t-variable, uniform in x).

$\partial_t \rightarrow i\tau \Rightarrow \partial_t = D_t^{\frac{1}{2}} H_t D_t^{\frac{1}{2}}$.

$$B(u,v) := \iint \left(H_t D_t^{\frac{1}{2}} u \cdot \overline{D_t^{\frac{1}{2}} v} + A\nabla_x u \cdot \overline{\nabla v}\right).$$

on $C_0^\infty(\mathbb{R}^{n+1})$.

$|B(u,v)| \leq C\|u\|_{\dot V}\|v\|_{\dot V}$. $\dot V = \overline{C_0^\infty(\mathbb{R}^{n+1})}^{\|\cdot\|_{\dot V}}$.

for $\|u\|_{\dot V}^2 = \|\nabla_x u\|^2 + \|D_t^{\frac12} u\|^2$.

Sobolev embedding $\dot V \subset L^{\frac{2(n+2)}{n}}(\mathbb{R}^{n+1})$.

This defines. $H: \dot V \to \dot V^*$,

$\mathrm{Re}\, B(u,u) \geq k\|\nabla_x u\|^2 \not\Rightarrow$ invertibility.

Nyström & independently (Auscher & ~~Egert~~ Auscher '15) $\leftarrow$ in context of max. reg.

$0<\delta<1$. $B_\delta(u,v) = B(u,(1+\delta H_t)v)$,

preserves domain,

$$\mathrm{Re}\, B_\delta(u,u) = \mathrm{Re}\, B(u,u) + \delta\, \mathrm{Re}\, B(u, H_t u).$$

$$= \iint A\nabla_x u \cdot \overline{\nabla_x u} + \delta\left(\|H_t D_t^{\frac12} u\|^2 + \mathrm{Re} \iint A\nabla_x u \cdot \overline{H_t \nabla_x u}\right).$$

δ small $\Rightarrow$ $\gtrsim_{\delta,k} \|u\|_{\dot V}^2$.

<u>Lax-Milgram</u>. $f \in \dot V^*$, $\exists!\, u \in \dot V$. $\forall v \in \dot V$.

$$B_\delta(u,v) = f((1+\delta H_t)v).$$

Thus, $Hu = f$.

$$B_\delta(u,v) = (Hu, (1+\delta H_t)v).$$

- $H : \dot{V} \to \dot{V}^*$ invertible
- $\mathrm{Re}\,(Hu, u) \geq 0.$
- $H : \dot{V} \to \dot{V}^*$ invertible.

$\mathcal{D}(H) = \{u \in L^2 : Hu \in L^2\}$

$(1 + H)^{-1}$ exists?

Uses $(u, (1 + \delta H_t)v) + B_\delta(u, v).$

- Key: Not looking at $[0, T]$ as usual parabolic theory, but whole of $\mathbb{R}$. It's this fact that contributes to invertibility.

Setup Pf for $\sqrt{H}$ estimates.

"First"-order approach: Harvest of results.

~~$Hu = 0$~~

Obtain $\sqrt{H}$ from the H^∞ f.c. of a Dirac-type operator.

$$P = \begin{pmatrix} 0 & \mathrm{div}_x & -D_t^{\frac{1}{2}} \\ -\nabla_x & 0 & 0 \\ -H_t D_t^{\frac{1}{2}} & 0 & 0 \end{pmatrix}. \quad M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

$$(MP)^2 = \begin{pmatrix} H & 0 & 0 \\ 0 & \boxed{\tilde{H}} & \\ 0 & & \end{pmatrix}, \quad \sqrt{(MP)^2} = \begin{pmatrix} \sqrt{H} & \\ & \boxed{\sqrt{\tilde{H}}} \end{pmatrix}.$$

2) Bisectoriality of MP and PM.

In fact, we can take. $M = \begin{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} & \begin{matrix} 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 \end{matrix} & 1 \end{pmatrix}$.

$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_{n+1}(\mathbb{C})$, , can ~~sta~~ replace by. +ve real-valued, but. don't know if $\mathbb{C}$ works.

~~str~~

$\underline{\text{Th}^{\underline{m}}}$. 1) PM & MP are bisectorial on $L^2(\mathbb{R}^{n+1}; \mathbb{C}^{n+2})$

2) They have H^∞-calculus on their range.

$$P^* = \begin{pmatrix} 0 & \mathrm{div}_x & H_t D_t^{\frac{1}{2}} \\ -\nabla_x & 0 & 0 \\ -D_t^{\frac{1}{2}} & 0 & 0 \end{pmatrix}. \quad \text{So} \quad P^* \neq P.$$

$$PM = \underbrace{(PN_\delta)}_{\text{s.a.}} (N_\delta^{-1} M). \qquad N_\delta = \frac{1}{\sqrt{1+\delta^2}} \begin{pmatrix} (1-\delta H_t) & & \\ & (1+\delta H_t) & \\ & & (\delta - H_t) \end{pmatrix}$$

$\boxed{\delta \text{ small.}}$.

the δ here is from $B_\delta(u,v)$ 'from the defining form.

$$h = \begin{bmatrix} h_\perp \\ h_{\parallel} \\ h_0 \end{bmatrix} \begin{matrix} \leftarrow \text{scalar} \\ \leftarrow \text{in } \mathbb{C}^n. \\ \leftarrow \text{scalar}. \end{matrix}$$

$$\mathcal{D}(P) = \{ h \in L^2 : \nabla_x h_\perp \in L^2, \quad H_t D_t^{\frac{1}{2}} h_\perp \in L^2$$
$$\operatorname{div} h_{\parallel} - D_t^{\frac{1}{2}} h_\theta \in L^2 \}.$$

$$N(P) = \{ h \in L^2 : h_\perp = 0 = -\operatorname{div} h_{\parallel} - D_t^{\frac{1}{2}} h_\theta = 0 \}.$$

$$\overline{\mathcal{R}(P)} = \{ h \in L^2 : \nabla_x h_\theta = H_t D_t^{\frac{1}{2}} h_{\parallel} \}.$$

$$L^2 = N(PM) \oplus \overline{\mathcal{R}(PM)}$$
$$= \cancel{\mathcal{R}(P)}\, N(P) \oplus \overline{\mathcal{R}(P)}.$$

<u>Sketch of proof of H^∞-calculus.</u>

$$\int_0^\infty \| \lambda MP (1 + \lambda^2 (MP)^2)^{-1} h \|_2 \frac{d\lambda}{\lambda} \simeq \|h\|_2^2 \quad \forall h \in \overline{\mathcal{R}(MP)}.$$

Enough to get $\lesssim$, and $h \in \mathcal{R}(MP) = M\mathcal{R}(P)$.

Ie,
$$\int_0^\infty \|\Theta_\lambda Pv\|^2 \frac{d\lambda}{\lambda} \lesssim \|Pv\|^2.$$

$$\Theta_\lambda = \lambda MP (1 + \lambda^2 (MP)^2)^{-1} M.$$

$T(b)$ argument $\to$ Reduction to a Carleson meas. estimate.

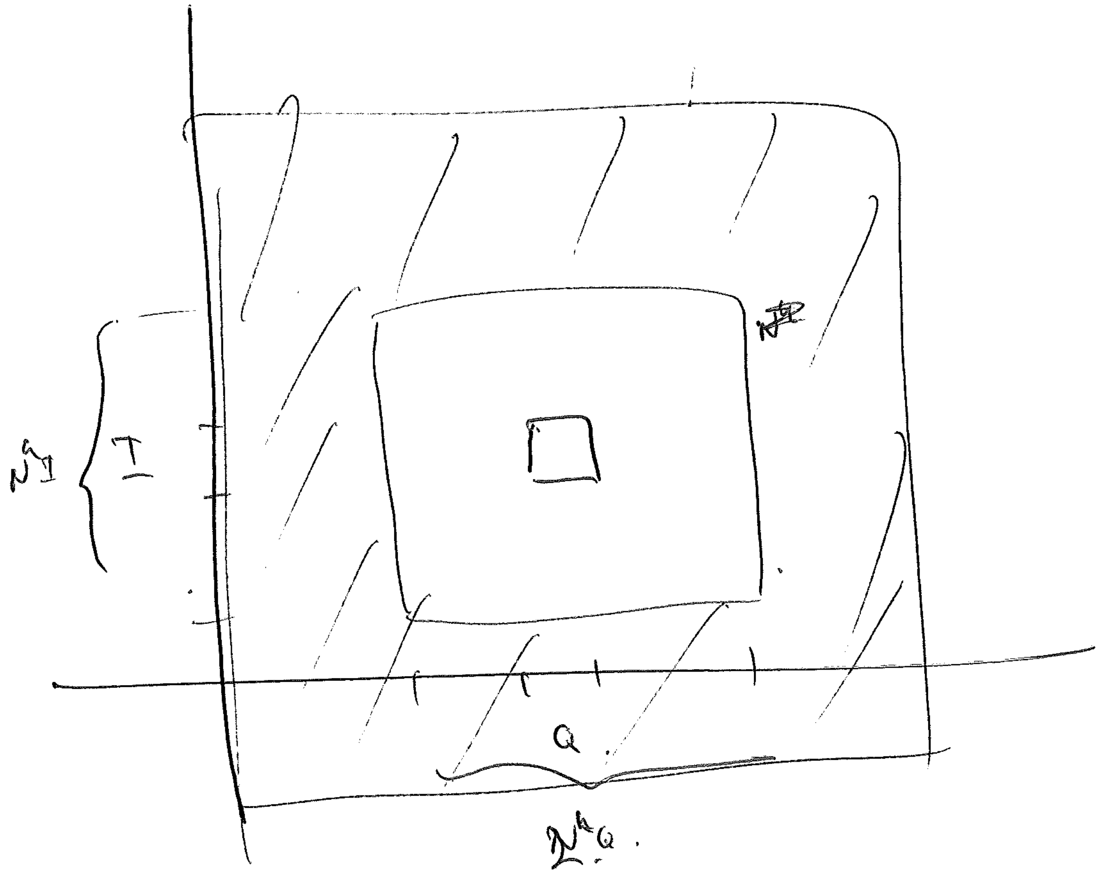
$\to$ Test functions to check CME.

<u>Reduction</u> Need off-diag decay,

$u \in C_0^\infty(\mathbb{R})$, $H_t D_t^{\frac{1}{2}} u = o(1+|t|^{-3/2})$ at ∞.

$$\iint_{Q\times I} |(1+i\lambda MP)^{-1} h|^2 . \quad h \in L^2, \text{ spt } h \subset (2^{k+1} Q \times N^{k+1} I) \setminus 2^k (Q \times N^k I).$$

$$\lesssim \left(2^{-km} + N^{-2k}\right) \iint_{C_k(Q\times I)} |h|^2 . \quad (*)$$



$$(1+i\lambda MP)^{-1} h_1 = (1+i\lambda MP)^{-1}(\eta h).$$

$$= (1+i\lambda MP)\, i\lambda M [P, \eta] \underbrace{(1+i\lambda MP)^{-1} h}_{L^2(\mathbb{R}^{n+1})}.$$

η spted away from $Q\times I$.

$[D_t^{\frac{1}{2}}, \eta] : L^2(\mathbb{R}, L^2(\mathbb{R}^n)) \to L^p(\mathbb{R}, L^2(\mathbb{R}^n))$. $2 < p < +\infty$.

win for $p > 2$.

<u>Lemma</u>. $(1 + i\lambda MP)^{-1} : L^p(L^2) \to L^p(L^2)$ $\left|\frac{1}{p} - \frac{1}{2}\right| << 1$.

ε in (A) is related to

→ reduction, need to estimate.

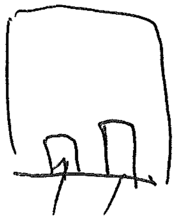
$$\iiint_{Q I \Theta} |\gamma_\lambda(x,t)|^2 \frac{dx\,dt\,d\lambda}{\lambda} \le c\,|Q \times I|.$$

$$\gamma_\lambda(x,t)(\xi) = \Theta_\lambda(x,t)(\xi). \quad \xi \in \mathbb{C}^{n+2} \text{ constant.}$$

$$\iint_{Q \times I} |\gamma_\lambda(x,t)|^2 \, dx\,dt\,\frac{d\lambda}{\lambda} \lesssim 1.$$

<u>Test functions</u>: $R = Q \times I$, $T(R) = (Q \times I)\,(0,r)$; b_R^ε subcollection of cubes $R' \subset R$.

$\sum |R'| \le (1 - \eta)\,|R|$. recur.



$T(R_1)$ $T(R_2)$.

Estimate $|\gamma_\lambda^\vee(x,t)|$ and $(\gamma_\lambda^\vee(x,t)\xi, \xi^*)$ by conical analysis.