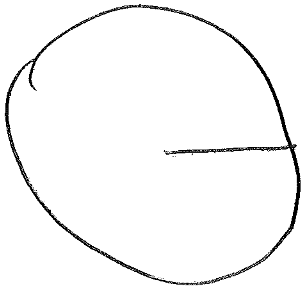
Rajala - Density of Sob. functions on Euclidean domains. 29/08/2016.

Question. Are Lipschitz/$W^{1,\infty}(\Omega)$ function in $W^{1,p}(\Omega)$.

①  slit removed. • Lipschitz not dense. • $W^{1,\infty}$ okay.

② Metric space setting: (X,d) complete separable m - locally finite Borel. $f \in W^{1,p}(X,d,m)$.
$\exists$ sequence f_i Lipschitz.

$$\lim_{i\to\infty} \int | |\nabla f_i| - |\nabla f|_{w,p}|^p \, dm = 0.$$

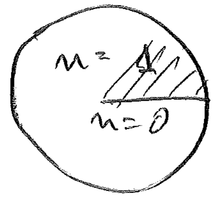
• But $\Omega \subset \mathbb{R}^n$ open subset, not complete.

• Defn of $W^{1,p}(\Omega) = \{u \in L^p(\Omega) : \nabla u \in L^p(\Omega)\}$.
ie, maximal domain.

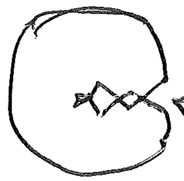
Classical results: $W^{1,\infty}(\Omega)$ dense in $W^{1,p}(\Omega)$ via fact that $C^\infty(\mathbb{R}^n)$ dense in $W^{1,p}(\Omega)$ and Ω hold

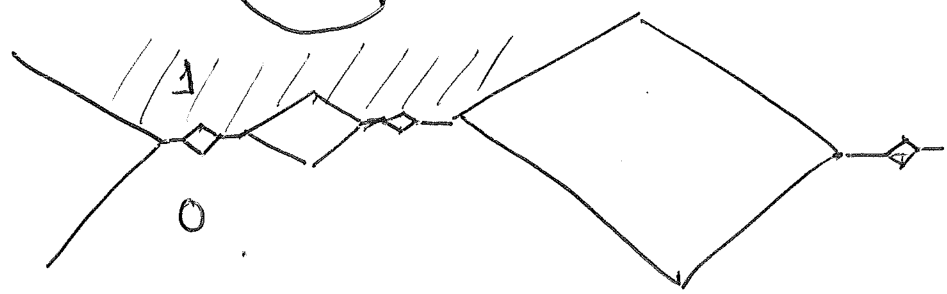
Particular case $C^\infty(\mathbb{R}^n)$ dense in $W^{1,p}(\Omega)$. when $\exists$ $E: W^{1,p}(\Omega) \to W^{1,p}(\mathbb{R}^n)$. This is an extension domain.

~~Ex~~

Non extension domain: 

$\Rightarrow$ any extension not abs cts.

Shark teeth domain:  $\leftarrow$ λ Cantor set

Question. 

Okay for some p, but not for others, i.e. $\dim(C_\lambda) \leq 2-p$. is sharp. Need $2\lambda^{2-p} \leq 1$.

<u>Uniform domain</u> $\Omega \subset \mathbb{R}^2$. $\forall\, x,y \in \Omega$, $\exists$ unit speed $\gamma \subset \Omega$
(I) $\ell(\gamma) \leq c\|x-y\|$. , (II) $\min(t, \ell(\gamma)-t) \leq c\, d(\gamma(t), \partial\Omega)$.

(Characterizes $W^{1,2}$ extendibility).

$2 < p < \infty$: Shvartsman ? 2010.
$1 < p < 2$: Rajala et. al. 2015 — but use now on the <u>outside</u> $\mathbb{R}^2 \setminus \Omega$

<u>Snowflake</u>: ex $W^{1,p}$ extension domain, but <u>not</u> locally graph of a function.

<u>Rmk</u>. Extendibility is sufficient, but not necessary. See "Segment domain".

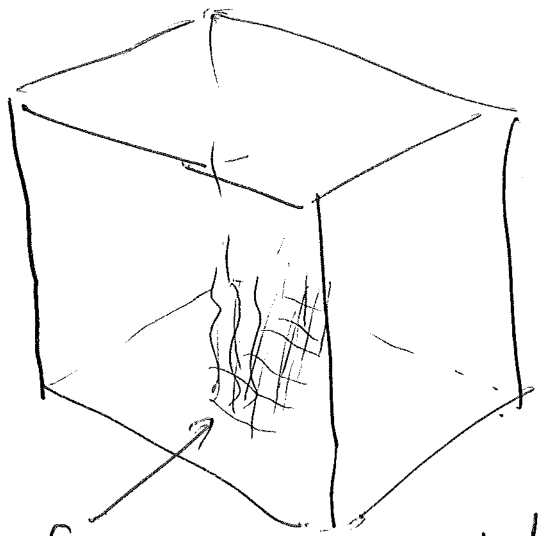
Q. planar simply connected Ω hold s.t. $W^{1,\infty}$ dense.

A. all! (Koskela-Zhang).

① $\exists$ compact $E \subset \mathbb{R}^2$. $\mathcal{L}(E)=0$. and removable for $W^{1,q}$, $p<q<\infty$. but not $1<q\leq p$.

$\Rightarrow \exists\ \Omega \subset \mathbb{R}^2$ s.t. $W^{1,q}$ not dense in $W^{1,p}$ $\forall q>p$.

② Higher dim:



E extended up into tube.

Set E: product of fat cantor set ~~dim < 1~~ and cantor set of dim < 1.

+ result of Shvartsman.

Quasihyperbolic metric: $d_{qh}(z_1,z_2) = \inf_\gamma \int_\gamma d(z,\partial\Omega)^{-1}\,dz$.

γ rectifiable curve b/w z_1, z_2.

Ω = Disc, d_{qh} is Poincaré model.

Results here are for δ-Gromov hyperbolic, no assumption on simple connectedness.

Pf Idea: Use δ-Gromov hyperbolicity:

Take Whitney cubes and p.o.u. to construct approximating function f_ε. Gromov hyperbolicity allows one to control "tentacles."