Ohta : Nonlinear geometric anal. in Finsler. 30/08/2016.

~~Fact~~ (M,F) C^∞-Finsler mfld, $F: TM \to [0,\infty)$.

(1) $F \in C^\infty$ (2). $F(cv) = cF(v)$.

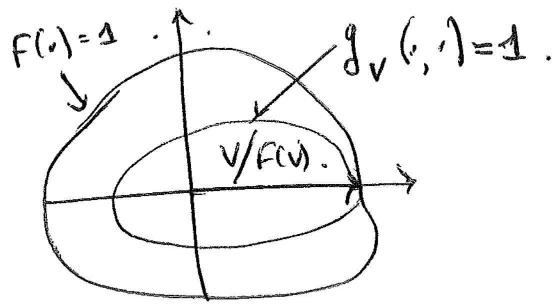
(3). $v \in TM \setminus \{0\}$, $\exists$ non sym metric.

$$g_{ij}(v) = \frac{1}{2} \frac{\partial^2 F^2}{\partial v^i \partial v^j}, \quad v = v^i \partial_i.$$

is positive definite (strong-convexity).

For each $v \in T_xM$, $g_v : T_xM \times T_xM \to \mathbb{R}$ inner product.

Approximation of Minkowski norm in the following sense.



Unit sphere of g_v tangent to unit sphere of $F(\cdot)$ upto second order.

- $d(x,y) = \inf_\gamma \int F(\dot\gamma(t))\, dt$. $\gamma(0)=x$, $\gamma(1)=y$.

  But $d(x,y) \neq d(y,x)$.

- geodesics as locally minimising curves of constant speed w.r.t. to d.

Curvature? Measure? lower or upper bound. Alex. $\to$ Riem.

Ricci curvature - no canonical measure. $\to$ start with arbitrary m and modify Ricci according to m.

Unweighted Ricci: Fix v, extend to C^∞ v. field V on Ω s.t. every integral curve is geodesic, Riem structure g_V on Ω

$Ric(v) = Ric_{g_V}(v)$ w.r.t. g_V (indep of V).

fix arbitrary m on M and $\mathrm{Ric}(v)$:
~~Φ~~ $m = e^{-\Phi}\,\mathrm{vol}_{g_v}$, η geodesic $\dot\eta(0) = v$.

$N \in (-\infty, 0) \cup [n, \infty]$, define Ric_N Bakry-Émery.

$N \in (n, \infty)$, $N' < 0$:

$$\mathrm{Ric}_n \le \mathrm{Ric}_N \le \mathrm{Ric}_\infty \le \mathrm{Ric}_{N'}.$$

- $N \in [n, \infty]$ now classical. Interest in $N < 0$.
  - → ~~E.~~ Milman isoperimetric.
  - → 0 curvature-dimension
  - → Wylie splitting theorem.
- $\mathrm{Ric}_N \ge k \iff$ Lott-Sturm-Villani's $CD(k, N)$.

<u>Non linear Lap</u>: $u: M \to \mathbb{R}$ diff at $x \in M$,

$\nabla u(x) \in T_xM$ as Legendre transform of $Du(x) \in T^*_xM$.

$$F^*(Du) = F(\nabla u).$$

For $u \in H^1_{loc}(M)$, non linear Lap $\Delta u = \mathrm{div}_m(\nabla u)$.
distributionally:

$$\int_M \varphi \Delta u \, dm = \int D\varphi(\nabla u) \, dm.$$

Δ is <u>nonlinear</u>. If u solves $\partial_t u_t + \Delta u_t = 0$.

$$\Rightarrow \quad u_t(x) \in C^{1,\alpha}(x, t).$$

<u>Bochner Inequality</u>: $u \in C^\infty(M)$ $N \in (-\infty, 0) \cup [n, \infty)$.

$$\Delta^{\nabla u}\left(\frac{F(\nabla u)^2}{2}\right) - D(\Delta u)(\nabla u) \ge \mathrm{Ric}_N(\nabla u) + \frac{(\Delta u)^2}{N}$$

pointwise on $\{\nabla u \ne 0\}$. weakly on M.

$\nabla^{\nabla u}$ linearised Laplacian $:= \mathrm{div}_{m}(\nabla^{\nabla u} f)$, $\nabla^{\nabla u} f = g^{ij}(\nabla u)\, \partial_i f\, \partial_j$.

<u>Rmk</u> Not Bochner inequality for $g_{\nabla u}$

This gives: host of inequalities, Log-Sobolev, Poincaré-Lichnerowicz, etc.

Some of these come from Cavalletti-Mondino in $F(-v) = F(v)$ (Reversible case), but localisation gives non-sharp results.

Γ_2 calculus gives sharper results.