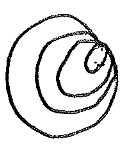
MMs w/ Aut. groups that are Lie. 01/09/2016.

Rmk $RCD^*(K,N)$ spaces have isometry groups that are Lie.

History:

- Riem mflds My-St. '39
- Alex[k] Fu-Ya '99.
- Ricci-limits. Ch-Co '99, Co-Na '12.

There are counter-examples:  Hawaiian ring.

Thm. $\exists$ metric meas. space (meas. pres.) (X,d,m) length mms.
Then both $Iso(X)$, $Iso_m(X)$ are Lie groups if:

[P1] m-a.e. Euclidean tangents. (existence & uniqueness).

w.r.t. Wasserstein: [P2] $\forall \mu_0, \mu_1 \in P^2(X)$. $\exists! \pi \in OptGeo(\mu_0,\mu_1)$ and π is given by a map.

Examples: Riem, Alex., $RCD^*(K,N)$-spaces $\Leftarrow$ Ricci limits. Alex. Cones.

$MCP(K,N)$ condition is not enough to get $Iso(X)$ is Lie

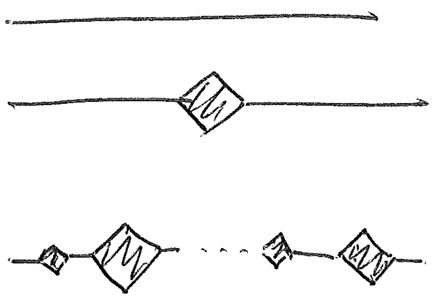
$\delta = 0$

$\delta = 1$

$\delta = \frac{1}{2}$

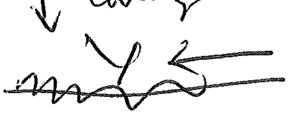
$\vdots$

$\delta = \frac{1}{n}$



$Y_n \in MCP(2,3)$.

$Iso(Y_n) = \mathbb{R} \; \{\pm 1\}$ covering



$Y \leftarrow$ also $MCP(2,3)$.

Open Q's: CD^* spaces, MCP non-branching.

Def. — (X,d,m) (X,d) comp. sep. metric space, m Borel and finite on bdd set.

— $(X_1,d_1,m_1) \approx (X_2,d_2,m_2)$,

$f: spt(m_1) \to X_2$ iso, $f_\# m_1 = m_2$ } m.p. iso.

$Iso(X)$ = usual metric iso, $Iso_m(X)$ = m-p iso } w/ compact-open top.

Apriori, not true that cpt-open top is good for $Iso_m(X)$.

$$Tan(X,p) := \{ (X_\infty, d_\infty, p_\infty) \xleftarrow[r_N \to 0]{pGH} (X, \tfrac{1}{r_N} d, p) \}$$

$$\mathcal{R} = \{ x \in X : Tan(X,x) = \{ (\mathbb{R}^n, d_{Euclid}) \} \ \exists N \}.$$

$m(X \setminus \mathcal{R}) = 0 \iff X$ has m-a.e. Euc. tangents.

$\mathcal{P}(X)$, $\mathcal{P}^2(X)$ = prob. meas. w/ finite second moments $\subset \mathcal{P}(X)$

$$\forall \mu_0, \mu_1 \in \mathcal{P}^2(X). \quad W_2^2(\mu_0,\mu_1) := \inf_\sigma \int_{X \times X} d^2(x,y) \, d\sigma(x,y).$$

(X,d) geo $\iff (\mathcal{P}^2(X), W_2)$ geo.

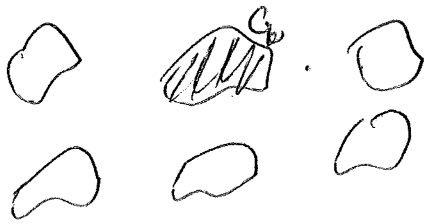
$\mu_0, \mu_1 \in \mathcal{P}^2(X)$, $OptGeo(\mu_0,\mu_1)$ = all measures geodesics in $Geo(X)$ s.t. $(e_0, e_1)_\# \pi$ minimizes in W_2.

If $\exists\, T: X \to \mathrm{Geo}(X)$ .. $\pi := T_{\#} m \in \mathrm{OptGeo}(\mu_0, \mu_1)$,
we say that π is induced by a map.

Def. G top group, G is a Lie group ← Generalisation

$\Leftrightarrow$ G_0 is a Lie group s.t. G/G_0 is discrete.

↑ C^∞ manifold.



Def. We say that G SSP if $\forall\, U_{id} \ni id$. $\exists\, id \neq V \subseteq U$

R1. $ISO(X)$. locally cpt w.r.t. c/o topp if X loc. compact.

R2. G locally compact top. group. then G Lie group iff G has no SSP.

Pf. [P2] $\Rightarrow$ fixed point set of all non-trivial isometry has measure zero.

[P1] $\Rightarrow$ $\exists$ non-trivial isometries with large fixed point sets i.e. positive measure.
+SSP in $ISO(X)$.

Actually stronger notion. $\forall x \in X$, $r \in \mathbb{R}$ $0 < r < 1$.
$\exists\, Id \neq \Lambda \leq ISO(X)$. $\forall g \in \Lambda$,
$m(\mathrm{Fix}(g) \cap B_r(x)) \geq m(B_r(x))$.

$ISOm(X)$ — loc compact + continuous metrics
— shows it is a closed subgroup $\Rightarrow$ Lie.