Cavalletti - Isoperimetric inequality for m.m. spaces. 29/08/2016.

Isoperimetry: Given X, what is minimal area needed to enclose a fixed volume $v>0$. Existence optimal shapes.

Levi-Gromov. (M,g) $\mathrm{Ric}_g \geq k g$, $k>0$.
$E \subset M$ C^∞ bdry ∂E. Then.

$$\frac{|\partial E|}{|M|} \geq \frac{|\partial B|}{|S|} \quad \text{LGI}.$$

(X,d,m), **outer Minkowski content:**

$$m^+(E) = \liminf_{\varepsilon \to 0^+} \frac{m(E^\varepsilon) - m(E)}{\varepsilon}, \quad E^\varepsilon = \{x : d(x,E) < \varepsilon\}$$

Isoperimetric profile function s.t. $\mathcal{I}_{(X,d,m)} : [0,1] \to \mathbb{R}^+$. largest

$$m^+(E) \geq \mathcal{I}_{(X,d,m)}(m(E)). \quad \forall E \subset X.$$

$$\mathcal{I}_{(X,d,m)}(v) := \inf\{m^+(E) : m(E) = v\}.$$

Rmk Classic rephrase:

$$\mathcal{I}(M, d_g, \mathrm{vol}/\mathrm{vol}(M)) \geq \mathcal{I}(S^n, d_{S^n}, \mathrm{vol}/\mathrm{vol}(S^n)).$$

(M,g,m) $m = \Phi \,\mathrm{vol}_g$.

CDD(k,N,D): • ~~supp~~ $\mathrm{spt}(m) \subset \Omega$ $\exists \Omega \subset X$ geodesically convex,

• $\mathrm{Ric}_{g,\Phi,N} = \mathrm{Ric}_g - (N-n) \frac{\nabla^2 \Phi^{1/N-n}}{\Phi^{N-n}} \geq k g.$

Milman: $k \in \mathbb{R}$, $N, D > 0$, $\exists \, \mathcal{I}_{k,N,D}$ s.t.
$\mathcal{I}_{M,g,m} \geq \mathcal{I}_{k,N,D}$.

Th^{m}. (C. – Mondino '15). $(X, d, m) \in CD_{loc}(K, N)$. and ess. non-branching, then ~~LG(Milman) ineq. holds~~.
LG M(ilman) iso. ineq. holds.

__Method of Pf__ Go from n-dim problem to 1-dim problem.

Payne – Weinberger '60: $K \subset \mathbb{R}^n$ bdd, open, convex, $f \in C^1(K, \mathbb{R})$.

$$\int_K f = 0 \Rightarrow \int_K f^2 \leq \frac{\operatorname{diam}(K)^2}{\pi^2} \int_K |\nabla f|^2 \quad \Big| \text{ Poincaré?}$$

Find hyperplane $H \subset \mathbb{R}^n$ s.t. $\int_{K \cap H^-} f = \int_{K \cap H^+} f = 0$.

then, keep going, by iteration, to get finer ~~part~~ partitions.

Gromov – V. Milman (87): Isoperimetric Inequality.
Kannan – Lovász – Simonovits ('95): Four function Th^{m}.

1-d localisation for mflds: Klartag ('14) using L^1-OT.

Th^{m}. $(M, d, m) \in CD(K, N)$ Convex Reimannian. $\int f \, dm = 0$.
$\exists \{M_\alpha\}_{\alpha \in Q}$ partition of M. Satisfying localisation properties.

- $m(A) = \int m_\alpha(A) \, q(\alpha)$ q measure over ~~on~~ Q
- ~~$\mathcal{S}$~~: hook it up localising properties.

Th^{m} Similar for $CD_{loc}(K, N) \Leftarrow$ C. Mondino '15.