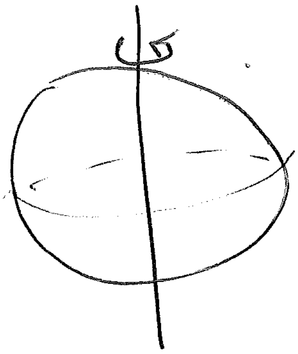
Mondello - Geom Anal on Strat. Spaces 30/08/2016.



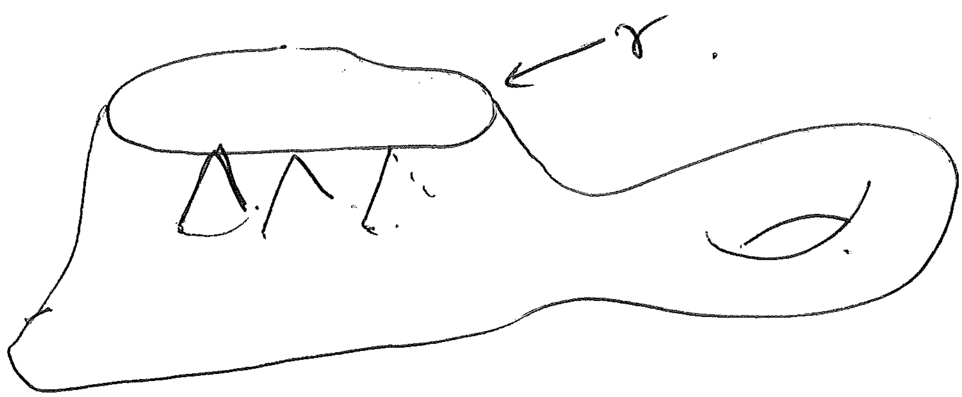
$\langle \pi \rangle$ (rotation by angle π).

$S^2/\langle \pi \rangle \longrightarrow$

Strat spaces:
- Quotients or limits.
- Generalize the notion of isolated conical singularities.
- Involved in topology; many people simple examples of strat. spaces.

2d cones:
- $S^1 \times [0,1]$ $\quad dr^2 + \left(\frac{\alpha}{2\pi}\right)^2 r^2 d\theta^2$.
- $\alpha \leq 2\pi$, Euclidean cone, curvature ≥ 0.
- $\alpha > 2\pi$, curvature ≤ 0.

- Non isolated conical singularities, say along a curve γ.



Manifold with simple edges:

$\Sigma^d \subset M^n$, $p \in \Sigma^d$, $U_p \simeq \mathbb{B}^d(\varepsilon) \times C(Z)$.

Z cpct $\nearrow$ cone.

Def^n. (X,d) compact metric space is called stratified if it admits the following decomposition

$$X = \Omega \sqcup \Sigma$$

Ω - regular, Σ - singular,

" ~~C^∞ manifold~~

- Ω C^∞ and dense in X.
- $\Sigma = \bigsqcup_{j=1}^{k} \Sigma^j$ Σ^j smooth manifold; $\dim \Sigma^j = j$.

  ~~Σ~~ no kinks
- $p \in \Sigma^j$, $\exists\, \mathcal{U}_p \simeq B^j(\varepsilon) \times C(Z)$.

  $\underbrace{Z \text{ stratified space}}_{\text{problem, singularities can be nested.}} \leftarrow$ LINK.

Metrics: Induction on n:

- g Riem on Ω,
- next to Σ^j $g = h + dr^2 + r^2 k_j$

  $\mathbb{R}^j$ Riem. k_j metric on Z.

n. Σ^{n-2}, $p \in \Sigma^{n-2}$ $\mathcal{U}_p \simeq B^{n-2}(\varepsilon) \times C(S^1)$.

$g = h + dr^2 + (\alpha/2\pi) r^2 d\theta$] Codim 2 condition

α angle of Z.

(2) Sob. spaces & curvature bounds:

- $W^{1,2}(X) = \overline{\mathrm{Lip}(X)}$.
- $C^\infty_c(\Omega)$ dense in $W^{1,2}(X)$.
- $\Delta_g \sim \mathcal{E}(u) = \int |du|^2 dv_g$.

} $W^{1,2}(X) \overset{i}{\hookrightarrow} L^p(X)$

$p \in [1, \frac{2n}{n-2}]$.

Thm (M, 14) (X^n, g) stratified space " $\mathrm{Ric}_g \geq n-1$ "

$\Rightarrow \forall f \in W^{1,2}(X)$,

$$\mathrm{Vol}_g(X)^{2/n} \|\varphi\|^2_{\frac{2n}{n-2}} \leq \|f\|_2^2 + \frac{4}{n(n-2)} \|d\varphi\|_2^2 .$$

Previous results: Thm (S. Ilias '80). same for cpct. mflds.

D. Bakry '94

(X, d, m) $m(X) = 1$, L generator of P_t diff. semigroup.

$A \subset \mathcal{D}(L)$. dense + other hypothesis,

Γ_1, Γ_2 carré du champ $\left(\begin{array}{l} (M,g) \ \Gamma(u,v) = \langle du, dv \rangle \\ \Gamma_2(u) = \Delta \frac{|du|^2}{2} + (d\Delta u, du) \end{array} \right.$

Def L satisfies $BE(n-1, n)$ if

$$\Gamma_2(f,f) \geq \frac{(Lf)^2}{n} + (n-1) \|df\|^2 .$$

(**Mfld:** $\mathrm{Ric} \geq n-1$: $\frac{\Delta |df|^2}{n} - \langle d\Delta f, df \rangle \geq \frac{(\Delta f)^2}{n} - (n-1)\Gamma(f,f)$)

Thm (Bakry) L satisfies $BE(n-1,n) \Rightarrow$ Sob. ineq. $\forall f \in \mathcal{D}(L)$

- Spectral gap.
- Bochner ineq. via integration by parts.

Pf of Sob. inequality.

(Def (X^n, g) is s.t. $\mathrm{Ric}_g \geq (n-1)$ if
(I) $\mathrm{Ric}_g \geq (n-1) g$ on Ω.
(II) angle along Σ^{n-2}. $\alpha \leq 2\pi$.

(I) Spectral gap: Th (M, 14) (X^n, g) smooth. $\mathrm{Ric} \geq (n-1)$
$\Rightarrow \lambda_1(\Delta_g) \geq n$.
angle condition & geometry of the links enter the picture here
• Study regularity of $\Delta\varphi = \lambda\varphi$.
• φ_ε cut-off fns: Bochner $\varphi_\varepsilon \to \varphi$ as $\varepsilon \to 0$.

(II). $r_0 \Delta u + u = u^{p-1}$.
$p < 2n/n-2$, r_0 best const in Sob. ineq.
$\varphi_\varepsilon u$. int by parts.
Bochner ineq. $\Rightarrow r_0 \leq \frac{4}{n(n-2)}$ ∎.

(III). Geometric consequences:
Th: (M, 15). diam $X \leq \pi$, and the follow are equiv!
$\lambda_1(\Delta_g) = n \Leftrightarrow \mathrm{diam}\, X = \pi \Leftrightarrow$ extremal fns for Sob. ineq.

Th. $\lambda_1(\Delta_g) = n \Leftrightarrow (X^n, g) \cong (\hat{X}^{n-1} \times (0,\pi), dt^2 + \sin^2 t\, \hat{g})$.
$(\hat{X}, \hat{g})$ smth. space.

Pf geodesic flow. to cones at singular pts.

Th $u \in W^{1,2} \cap L^2(X)$. $\tilde{g} = e^{2u} g$, $Y(X, [g]) = \inf_{\tilde{g} \in [g]} \int_X \mathrm{Scal}_{\tilde{g}}\, dv_{\tilde{g}}$ $\geq V^{\frac{2}{n}} \frac{n(n-2)}{4}$

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