Kaj Nyström .- 2nd order parabolic w/ complex coefficients  13/09/2010

$$\mathcal{H}u = (\partial_t + \mathcal{L})u = \partial_t u - \mathrm{div}_x A(x,t)\nabla_x u = 0.$$

## Parabolic Problems:

$$(X,t) = (x_0, x, t) = (\lambda, x, t).$$

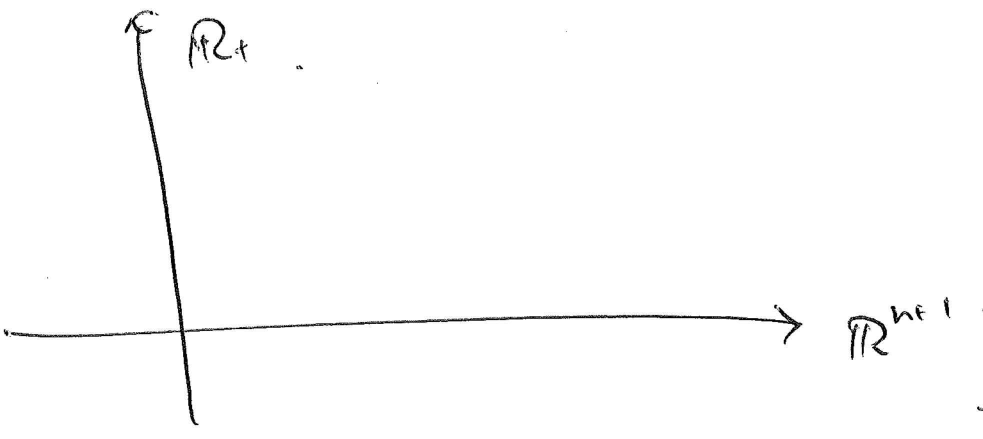
defining upper half space.  sometimes $x_0 = \lambda$.

$$\nabla_X = \nabla_{\lambda,x} = (\partial_\lambda, \nabla_x). \qquad \nabla_x = \nabla_\parallel$$

$$\mathrm{div}_X = \mathrm{div}_{\lambda,x} = (\partial_\lambda, \mathrm{div}_x). \qquad \mathrm{div}_x = \nabla_\parallel.$$

$$\mathcal{H}_\parallel = \partial_t + \mathcal{L}_\parallel = \partial_t - \mathrm{div}\, A\nabla_x$$

$$\mathbb{R}^{n+2}_+ = \{(X,t) : (x_0, x, t) \in \mathbb{R}^{n+1} \times \mathbb{R},\ x_0 > 0\}.$$




Parabolic problem lives here

Weak sol$^{n}$ $u \in L^2_{loc}(\mathbb{R}, W^{1,2}_{loc}(\mathbb{R}^{n+1}_+))$.

$$\langle A \nabla_{\lambda,x} u, \nabla_{\lambda x} \varphi \rangle + \langle u, \partial_t \varphi \rangle.$$

$\Rightarrow \partial_t u \in L^2_{loc}(\mathbb{R}, W^{-1,2}_{loc}(\mathbb{R}^{n+1}_+))$.

$\dot{H}^{\frac{1}{2}}(\mathbb{R}) \sim \| D_t^{\frac{1}{2}} \cdot \|_2$.

$\dot{E}(\mathbb{R}^{n+2}) \sim \| \nabla_{\lambda,x} \cdot \|_2 + \| H_t D_t^{\frac{1}{2}} \cdot \|_2$.

Reinforced weak sol$^{n}$: $u \in \dot{E}_{loc} := \dot{H}^{\frac{1}{2}}(\mathbb{R}; L^2_{loc}(\mathbb{R}^{n+1}_+)) \cap L^2_{loc}(\longrightarrow)$.

globally $H^{\frac{1}{2}}$ in time, but finite in space.

Hidden Coercivity:

$$a_\delta(u,v) = \langle A \nabla_{\lambda,x} u, \nabla_{\lambda x}(1 + \delta H_t) v \rangle$$
$$+ \langle H_t D_t^{\frac{1}{2}} u, D_t^{\frac{1}{2}}(1 + \delta H_t) v \rangle.$$

$\Rightarrow \operatorname{Re} a_\delta(u,u) \geq (k - C\delta) \| \nabla_{\lambda x} u \|^2 + \delta \| H_t D_t^{\frac{1}{2}} u \|^2.$

Ellipticity constants of $A$.

Use Lax-Milgram to obtain weak solutions. $\square$

~~Again with~~

## ~~Re~~ Maximal accretivity & selfadjointness:

Using same methods for hidden coercivity, get $H_{\parallel}$ to be max. accretive, but also

$$\|(\theta + H_{\parallel})^{-1} u\| \leq \frac{1}{|\operatorname{Re}\theta|} \|u\| \quad \text{for } \operatorname{Re}\theta < 0.$$

## Introduce associated F.O.S.

$$D_I u(\lambda, x, t) = \begin{bmatrix} \partial_\lambda u(\lambda, x, t) \\ \nabla_x u(\lambda, x, t) \\ H_t D_t^{\frac{1}{2}} u(\lambda, x, t) \end{bmatrix} = \begin{bmatrix} F_\perp \\ F_\parallel \\ F_\theta \end{bmatrix}.$$

$$P = \begin{pmatrix} 0 & \operatorname{div}_x & -D_t^{\frac{1}{2}} \\ -\nabla_x & 0 & 0 \\ -H_t D_t^{\frac{1}{2}} & 0 & 0 \end{pmatrix} \leftarrow \text{contains non-local derivatives, non-s.a.!}$$

Flat system. Given $A$, consider instead

$$D_A u(\lambda, x, t) = \begin{pmatrix} \partial_{\nu_A} u(\lambda, x, t) \\ \nabla_x u(\lambda, x, t) \\ H_t D_t^{\frac{1}{2}} u(\lambda, x, t) \end{pmatrix}.$$

$$\|D_A u\|^2 \sim \|\nabla_{\lambda, x} u\|^2 + \|H_t D_t^{\frac{1}{2}} u\|^2.$$

Then, get system $\partial_\lambda F + PMF = 0$,

$$M = \begin{pmatrix} \hat{A}_{\perp\perp} & \hat{A}_{\perp\parallel} & 0 \\ \hat{A}_{\parallel\perp} & \hat{A}_{\parallel\parallel} & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Bisectoriality of PM:

$$U_\delta = \frac{1}{\sqrt{1+\delta^2}} \begin{pmatrix} 1-\delta H_t & & \\ & 1+\delta H_t & \\ & & \delta - H_t \end{pmatrix}.$$

$$PM = \underbrace{(PU_\delta)}_{\text{s.a.}} \underbrace{(U_\delta^{-1} M)}_{\delta > 0 \text{ small} \Rightarrow \text{accretive}}.$$

Diff between t indep / dep:

- Poincaré local / global.

  $(\nabla_x u, \partial_t u)$    $(\nabla_x u, D_t^\alpha u, H_t D_t^\alpha u)$.

- Carleson meas:
  - $\|(1+\lambda^2 H_{\|})^{-1} \operatorname{div}_{\|} A_{\|}\|^2 \lambda\, dx\, dt\, d\lambda$.

  vs. $PM, (MP)$.

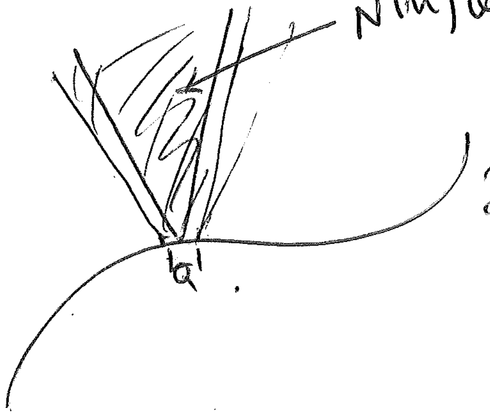
- Off-diag estimates. classical (elliptic) vs. weaker formulation. $(1+i\lambda PM)^{-1}$ on cylinders stretched in time.

- Th $th^{\perp}$: test functions close to elliptic construction vs handling $H_t D_t^{\frac{1}{2}}$.

Martin Dindoš. – Bdry val. problems for 2nd order elliptic eqs satisfying Carleson cond. 14/07/2016.

Model $-\mathrm{div} A \nabla$.

$N(u)(Q) = \sup_{x \in \Gamma(Q)} |u(x)|.$ $\Leftarrow$



$\tilde{N}(u)(Q) = \cancel{\sup_{x\in\Gamma(Q)}}$ – modified via $L^2$ averages

$\tilde{N}$ better suited in the $\mathbb{C}$ or systems case.

$L^p$ Dirichlet: $f \in L^p(\partial\Omega)$, $Lu = 0$ with $u|_{\partial\Omega} = f$,

$$\|N(u)\|_{L^p(\partial\Omega)} \lesssim \|f\|_{L^p(\partial\Omega)}.$$

Depends only on $L$, $p$ and $\mathrm{Lip}(\partial\Omega)$.

Neumann:
$$\begin{cases} Lu = 0 & \text{in } \Omega. \\ \underbrace{A\nabla u \cdot \nu}_{\text{conormal derivative.}} = f & \text{in } \partial\Omega. \end{cases}$$

$$\|\tilde{N}(\nabla u)\|_{L^p(\partial\Omega)} \lesssim \|f\|_{L^p(\partial\Omega)}.$$

Remark Even in scalar case, $u \in C^\infty$ but $\nabla u \in$ ?
So, need $\tilde{N}$.

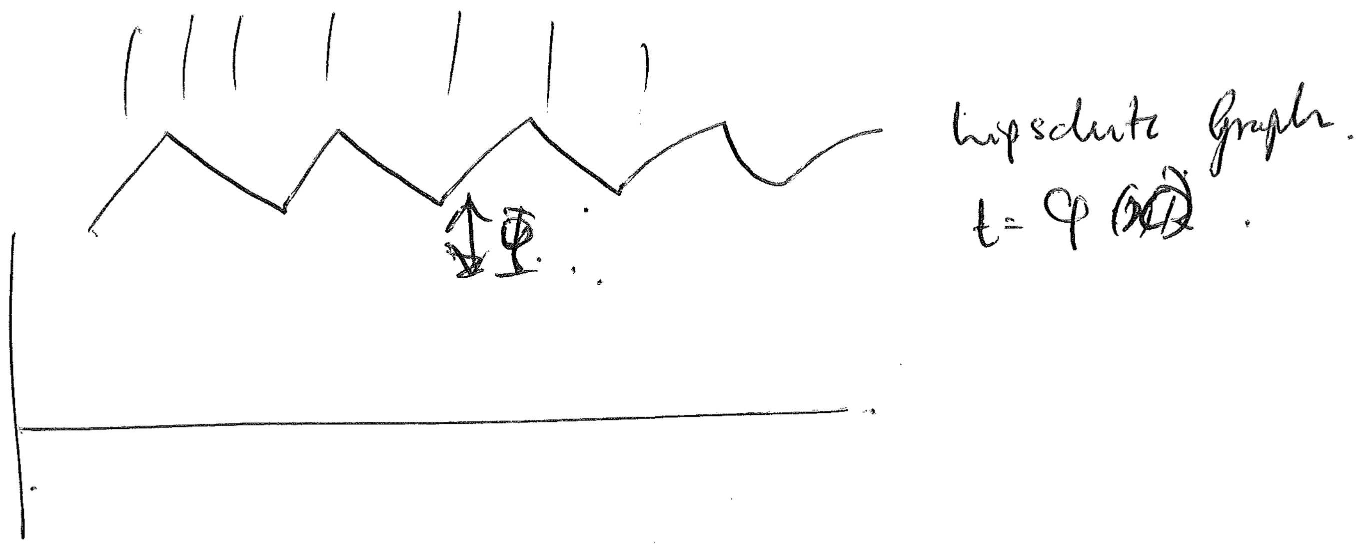
p-Regularity:
$$Lu = 0 \quad \text{in } \Omega.$$
$$u|_{\partial\Omega} = f \text{ on } \partial\Omega.$$

$$\|\tilde{N}(\nabla u)\|_{L^p(\partial\Omega)} \lesssim \|\nabla_T f\|_{L^p(\partial\Omega)}.$$

Th$^m$. $\exists$ $A \in L^\infty$ on unit disc s.t.
$D_p$, $N_p$, $R_p$ are not solvable for any
$p \in (1, \infty)$.

$\Rightarrow$ Solvability requires extra assumptions on coefficients $A$.

Carleson Condition : motivation.




$$\Phi : \mathbb{R}^n_+ \longrightarrow \{ X = (x,t) : t > \varphi(x) \}.$$

Define $\Phi$ with convolution $(\varphi_t)_{t>0}$ to get a little smoothing. Carleson measure density:

$$\delta(x)^{-1}\left(\operatorname{osc}_{B(x,\delta(x)/2)} a_{ij}\right)^2 \qquad \text{(Density)}$$

$\mu$ Carleson: $\quad \mu(B(Q,r)\cap\Omega) \leq C\,\sigma(B(Q,r)\cap\partial\Omega))$

Kenig - Pipher: '01: If (Density) is a Carleson measure density on Lipschitz dom $\Omega$, then $(D_p)$ solvable for some large $p<\infty$.

M.D - Pipher - Petermichl '07: $\forall\, p\in(1,\infty)$ $\exists\, c=c(p)>0$ s.t. Carleson norm $< c(p)$ and $\operatorname{Lip}(\Omega)<c(p)$.
$\Rightarrow$ $(D_p)$ solvable.

Main Thm $1<p<\infty$, $\Omega$ Lip. $\operatorname{Lip}(\Omega)=L$, and (Density) is a density of a Carleson measure on all Carleson boxes of size at most $r_0$ with norm $C(r_0)$. Then, $\exists\ \varepsilon=(\Lambda,n,p)>0$ such that if $\min\{L, C(r_0)\}<\varepsilon$.
Then, $(R_p)$ and $(N_p)$ solvable. $f\in L^p(\partial\Omega)$.

Neg problem: solving $p=2$ enough.

M.D.–Kirsch 2012: $(R_1)$ solvable. Then $\forall p \in (1,\infty)$.

$(R_p) \Leftrightarrow (D^*_{p'})$.

$(R_2) \Rightarrow (R_1)$. ( $(R_1)$ in Hardy–Sobolev space ).

$(D^*_p)$ is Dirichlet for $L^*$.

<u>$p=2$ and sq. function:</u>

$$\| S(\nabla u) \|_{L^2} \lesssim \text{bdry data} + \varepsilon \| N(\nabla u) \|_{L^2}.$$

<u>and</u> $$\| S(\nabla u) \|_{L^2} \gtrsim_\varepsilon \| N(\nabla u) \|_{L^2}.$$

$$S(v) = \left( \int_{\Gamma(a)} |\nabla v(x)|^2 \, \delta(x)^{2-n} \, dx \right)^{\frac{1}{2}}.$$

For Neumann problem: no analogous result ~~for~~ like M.D–Kirsch. So, do induction for $p$ integer. then interpolate.

<u>Open problem:</u> (Density) ~~the~~ is a density for Carl, no control on smallness of it, but solvable for <u>some</u> $p > 1$.

<u>Hint:</u> by MD–Kirsch, show $(R_{1+\varepsilon})$ for some small $\varepsilon > 0$.

__Systems__ $Lu = \operatorname{div} A\nabla + B\nabla.$ , $A, B$ tensors on $\mathbb{R}^n$:

Assume $A, B$ strongly elliptic & bdd.

__Outline:__
- $L^2$ Dirichlet (for symm.).
- $L^2$ Reg. (good progress).
- $L^p$ Dirichlet. $\left(2-\varepsilon < p < \frac{2(n-1)}{n-3} + \varepsilon \text{ in progress}\right)$
- Neumann, Reg?

__Scalar eqn, $\mathbb{C}$-coeff:__ – think about it as a $\mathbb{R}$-system, but skew-symm., cannot apply.
Stuff from before.

__New idea:__ Direct method for $L^p$ solvability:
~~Cap~~ Concept of $L^p$ dissipativity of Cialdea & Maz'ya
in the context of parabolic PDE:
$L$ $L^p$ dissipative. $u_t - Lu_x = 0$ satisfies:

$$\|u(t)\|_{L^p} \leq \|u(0)\|_{L^p} \qquad t \geq 0.$$

(Constant $=1$). C+M have algebraic condition.

$p$-adapted ellipticity:

$$\langle \operatorname{Re} A\lambda, \lambda\rangle + \langle \operatorname{Re} A\eta, \eta\rangle + \left\langle\left(\sqrt{\tfrac{p'}{p}}\operatorname{Im} A - \sqrt{\tfrac{p}{p'}}\operatorname{Im} A^t\right)\lambda, \eta\right\rangle \simeq |\lambda|^2 + |\eta|^2.$$

Condition comes from stability and $p = 2$,

$\exists \; p_0 \in [1, 2)$ s.t. $p$-diss. holds for $p \in (p_0, 2+\delta)$