Charles Batty – Proving hol. semigroups.

TFAE : ⓐ $-A$ generates hol. semigroups.
ⓑ A sectorial with $\omega_A < \pi/2$.

$(\Leftarrow)$. $T(z) = \exp(-zA) = \frac{1}{2\pi i}\oint_\gamma e^{-z\xi} R_A(\xi)\, d\xi$.

$T \in C^\infty((0,\infty), X)$.

$(\Rightarrow)$. Bdd semigroup means $T(t_1+t_2) = T(t_1)T(t_2)$.

$\| \cancel{\lim} \, T(t)x - x \| \to 0$ as $t \to 0$.

$$(\lambda - A)^{-1} x = \int_0^\infty e^{-\lambda t} T(t) x \, dt \quad (x \in X, \lambda \in \mathbb{C} = \Sigma_{\pi/2}).$$

But. Eg C_0-semigroup $T : [0,\infty) \to \mathcal{L}(X)$,

and $-A$ generates a bdd C_0 semigroup

$\Rightarrow$ A sectorial with $\omega_A \le \pi/2$.

But not strictly

Subordination.

$\mu_{t_1} * \mu_{t_2} = \mu_{t_1+t_2}$ $\quad (\mu_t)_{t \ge 0}$ convolution semigroup of sub-probability measures.

$\exists !\, f : [0,\infty) \to (0,\infty)$ s.t.

$$\int_0^\infty e^{-s\lambda}\, d\mu_t(s) = e^{-tf(\lambda)} .$$

These are known as <u>Bernstein functions</u>.
and/or <u>Laplace exponents</u>:

Take formula, replace λ by an A, operator.

<u>Bernstein</u>: $f : (0,\infty) \to (0,\infty) \Leftrightarrow f \in C^\infty$ and

$$(-1)^{n-1} f^{(n)}(t) \geq 0 . \qquad (n \geq 1,\ t > 0) .$$

Equivalently, $\exists$ ν pos. meas, $a, b > 0$ s.t.

$$\begin{cases} f(\lambda) = a + b\lambda + \int_0^\infty (1 - e^{-s\lambda})\, d\nu(s) . \\ \int_0^\infty \frac{s}{1+s}\, d\nu(s) < \infty . \end{cases}$$

By this formula, f extends to holomorphic $f : \mathbb{C}_+ \to \mathbb{C}_+$.

So now, replace λ (or z) by op. A, get $f(A) = aI + bA + \int_0^\infty (1 - e^{-sA})\, d\nu(s)$.

So unbdd.

If $-A$ gens bdd C_0 semigroup,

$$T_f(t)x = \int_0^\infty T(s)x \cdot d\mu_t(s).$$

Define $\quad f_0(A)x = ax + bA + \int_0^\infty (x - T(s)x)\, d\nu(s).$

$x \in \mathcal{D}(A).$

$\Rightarrow \quad -f(A) = \overline{f_0(A)}$ ~~generator~~ ; $\quad -f(A)$ generator for

T_f (Bochner '49, etc.).

<u>Q</u>. T bdd hol semigroup, f Bernstein, then is T_f also hol?

<u>Fun. Calc.</u>: $\theta > \omega_A$, A sectorial, dense range $\Rightarrow$ A injective and $A^{-1} : \mathcal{R}(A) \to X$. sectorial of same angle.

$f : \Sigma_\theta \to \mathbb{C}$ defines $f(A)$ as a closed operator.

<u>Thm</u>. A sectorial $\omega_A < \pi/2$, f Bernstein, $f(A)$ sectorial with ~~$~~ $\omega_{f(A)} \leq \omega_A$.

Q. A sectorial, f hol, when is f(A) sectorial?

Q1. for which f is ~~B(~~ $\omega_{f(A)} \le \omega_A$?

Q2. for which A is f(A) sec. ?

Q1. closed under reciprocals, sums etc.

↑ f should be: · hol on $\mathbb{C}_+ \longrightarrow \mathbb{C}_+$.
· map $(0,\infty) \longrightarrow (0,\infty)$.

Called positive real function. "NP_+ functions"

NP_+ closed under sums, pos. scalar multiples, reciprocals, compositions.

maps $\Sigma_\theta \longrightarrow \Sigma_\theta$.

represented by: $$f(z) = \int_{-1}^{1} \frac{2z}{(1+z^2)+t(1-z^2)}\, d\mu(t).$$

for ∃ pos. Borel measure μ on $[-1,1]$.

Q2 has thm which says that f(A) sectorial $\omega_{f(A)} \le \omega_A$ iff A has bdd hol. f.c. (on Hilbert space).

~~[illegible]~~

Way to proceed :

$$(\lambda + f(z))^{-1} = \frac{1}{\lambda+f(\infty)} + \frac{\alpha}{\pi}\int_0^\infty \frac{\operatorname{Im} f(te^{i\pi/\alpha})\,t^{\alpha-1}}{(\lambda+f(e^{i\pi/\alpha}))(\lambda+f(te^{-i\pi/\alpha}))(t^{\alpha}+z^{\alpha})}\,dt .$$

But can't get estimated nicely, maybe improper integral, so impose condition on f.

$f(\infty)$ and $f(0+)$ need to exist, if it doesn't, $\exists A$ s.t. $f(A)$ <u>not</u> sectorial.

This condition is closed under many properties.

For instance $g_\alpha(z) = f(z^\alpha)^{\frac{1}{\alpha}}$ with f Bernstein.

Ie, $\sqrt{z(1-e^{z})}$, ~~z~~