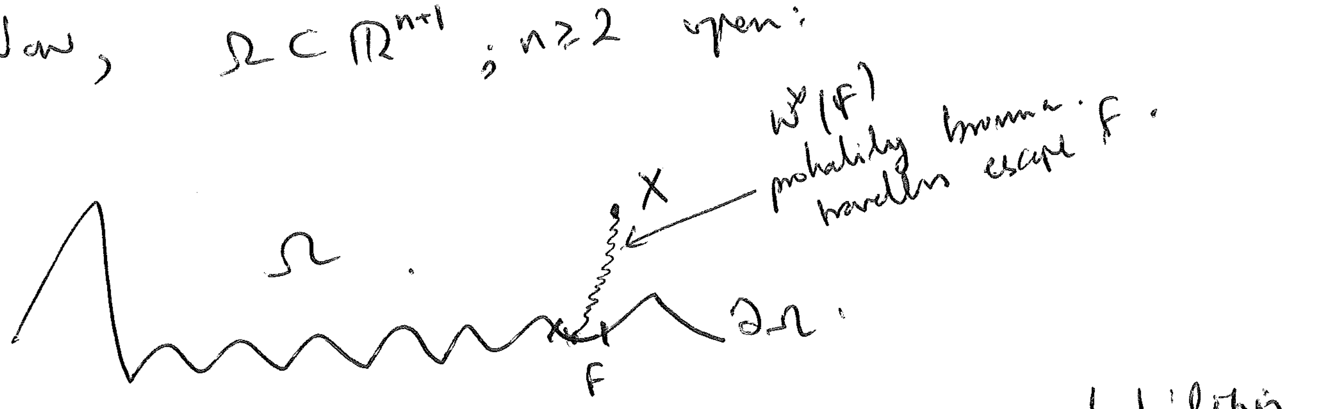
José María Martell   Free boundary results.   14/09/2016.

<u>Old Thm of Riesz 1916:</u> $\Omega \subset \mathbb{C}$ simply connected with rectifiable boundary $\Rightarrow$ harmonic measure $\omega \ll \sigma = \mathcal{H}^1|_{\partial\Omega}$.

1990: Result can fail without some topology.

• Now, $\Omega \subset \mathbb{R}^{n+1}$, $n \geq 2$ open:




Harm. measure. $\{\omega^X\}_{X\in\Omega}$ family of probabilities.

$$u(X) = \int_{\partial\Omega} f(x)\, d\omega^X(x). \quad \text{solves} \quad \begin{cases} Lu = 0 \text{ in } \Omega. \\ u|_{\partial\Omega} \text{ "=" } f \in C_c(\partial\Omega). \end{cases}$$

• Surface ball $\Delta(x,r) = B(x,r) \cap \partial\Omega$. $x \in \partial\Omega$.

• $\sigma = \mathcal{H}^n|_{\partial\Omega}$.

• $\partial\Omega$ ADR $\rightsquigarrow$ $\sigma(\Delta(x,r)) \approx r^n$, $x \in \partial\Omega$.

Rectifiability: $\bigcup_0^\infty$ union of Lipschitz images $\sigma$-a.e.

Uniform Rect: ADR & Big pieces of Lipschitz images.

E UR $\iff$ E ADR + All <u>nice SIO's</u> are bdd on $L^2(E)$

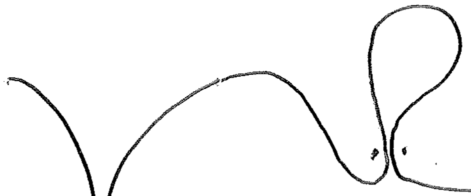
any convolution op with odd kernel:

1

Openness $\xrightarrow{\text{quantitative version}}$ Corkscrew condition Jerison–Kenig.

Path connectedness $\longrightarrow$ Harnack chain condition.

(connect path by balls, proportional to distance to $\partial\Omega$).



no Harnack chain.

no corkscrew.

$\Omega$ CAD $:=$ $\begin{cases} \text{Interior corkscrew + Harnack chain.} \\ \text{Exterior corkscrew.} \\ \partial\Omega \text{ ADR.} \end{cases}$

1-sided CAD — no Ext condition.

1990 Jerison & David:

$\Omega \subset \mathbb{R}^{n+1}$ CAD $\Rightarrow$ $w \in A_\infty(\sigma)$.

$\left( \frac{w(F)}{w(\Delta)} \lesssim \left(\frac{\sigma(F)}{\sigma(\Delta)}\right)^{\theta} \quad F \subset \Delta. \right.$

Rectifiability: $\Omega$ CAD $\Rightarrow$ $\partial\Omega$ UR.

<u>Thm</u>. ~~$\partial\Omega$ UR $\Leftrightarrow$~~ $\Omega \subset \mathbb{R}^{n+1}$ 1-sided CAD.

$\partial\Omega$ UR $\Leftrightarrow$ $\Omega$ CAD $\Leftrightarrow$ $w \in A_\infty$:

<u>Other operators?</u> — Condition about $A$.

Sketch of Pf for $\Delta$: W.T.S. $w \in A_\infty \Rightarrow$ Ext. cond.

$\partial\Omega$ ADR $\to$ $\mathbb{D}$ - dyadic grid (say Christ).

$\mathcal{B} = \mathcal{B}(C_0) = \{Q \in \mathbb{D} : C_0\text{-ext corkscrew fails for } Q\}$.

Goal: packing condition.

$$\sup_{Q \in \mathbb{D}} \frac{1}{\sigma(Q)} \sum_{Q' \in \mathcal{B} \cap \mathbb{D}_Q} \sigma(Q') =: M < \infty$$

$\Rightarrow Q \in \mathbb{D} \Rightarrow \exists Q' \notin \mathcal{B} \cap \mathbb{D}_Q$ s.t. $2^{-M} \ell(Q) \le \ell(Q') \le \ell(Q)$

$Q'$ has a $C_0$ ext cks $\Rightarrow$ $Q$ has a $C_0 2^{-M}$ ext cks.

Pf of packing John-Nirenberg reduction: Suffices to show

$\forall Q_0 \in \mathbb{D}$, $\exists \mathcal{F} = \{Q_i\}_i \subset \mathbb{D}_{Q_0}$ pairwise disjoint.

Satisfies ____________________ ?

Bourgain + Normalisation + subdivide & do stopping time argument.

(a) $1 \le \frac{w(Q_0)}{\sigma(Q_0)} \le C_0$. (b) $w \in A_\infty$.

(a) Stopping for too big and (b) for too small.

$L = \operatorname{div} A\nabla$. $A$ real, symm. $\omega_L$ elliptic.

~~Show: $\omega_L \in A_\infty \Rightarrow$~~

$\Omega \subset \mathbb{R}^{n+1}$ CAD, $\delta(x) = \operatorname{dist}(x, \partial\Omega)$.

Carleson cond.

$|\nabla A|\delta \in L^\infty(\Omega)$ + some other condition.

<u>Goal:</u> Find $A$ s.t. $\omega_L \in A_\infty \Rightarrow \partial\Omega$ UR or $\Omega$ CAD.

In this setting, we get. $(RH_p) \Leftrightarrow L^{p'}$ Dirichlet Solvability $(*)$

$$\left( \fint_\Delta k^p \, d\sigma \right)^{\frac{1}{p}} \leq \fint_\Delta k \, d\sigma = \frac{\omega(\Delta)}{\sigma(\Delta)}.$$

$\omega \ll \sigma$, $k = \frac{d\omega}{d\sigma}$.

$$A_\infty = \bigcup_{p>1} RH_p.$$

<u>Perturbation:</u> $L_0$ good op, want to perturb to $L$.

$$a(x) = \sup_{Y \in B(x, \delta(x)/2)} |A_0(Y) - A(Y)| \rightarrow d\mu(x) = \frac{a(x)}{\delta(x)} dx$$

<u>Thm</u>.

$\Omega$ 1-sided CAD

① Carleson: $\sup_{\Delta \subset \partial\Omega}$ ______ . "Carleson perturbation".

② ______.

$\Rightarrow (*)$.

③