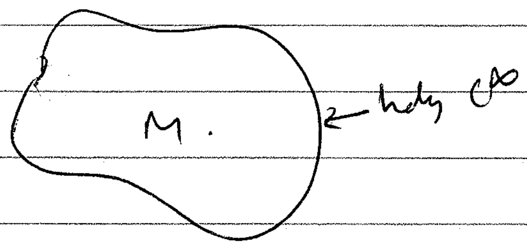
Rafe Lecture 8.

15/10/2012.

Elliptic Boundary Problems.



P diff. operators

$Pu = f$

$B(\gamma u) = h$.

$\gamma u = \begin{pmatrix} u_0 \\ \vdots \\ u_{m-1} \end{pmatrix}$, $\left(\frac{1}{i}\nu\right)^j u|_{\partial\Omega}$.

normal.

$u_0 \in H^{m-\frac{1}{2}}(\partial\Omega)$, $u_j \in H^{m-j-\frac{1}{2}}(\partial\Omega)$.

<u>Note</u>: We let P be a diff op., but why not Pseudo? Interestingly, there is <u>not</u> a good theory, pseudo-diff. ops don't respect boundary well.

(P, B) is an elliptic boundary problem if

$$B(\gamma u) = \begin{pmatrix} \sum_{j=0}^{m-1} B_{1j} u_j \\ \vdots \\ \sum_{j=0}^{m-1} B_{rj} u_j \end{pmatrix}.$$

$B_{ij} \in \Psi^*(\partial M)$, $\quad u_k \in H^{m-k-\frac{1}{2}}$; $B_{ik} u_k \in H^{s_i}$.

$\deg B_{ik} = m-k-\frac{1}{2}-s_i$.

<u>Goal</u>. $f \in L^2$, $h_i \in H^{s_i}(\partial\Omega)$, want to find (essentially) unique sol^{n}.

Eg. $\Delta u = f$, $u_0 = h_0$ (Dirichlet).
$\Delta u = f$, $u_1 = h_1$ (Neumann).

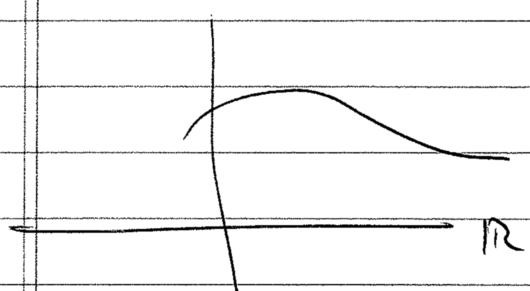
Background information on Sobolev spaces.

$$H^m(M) = \{u|_M : u \in H^m(\tilde{M})\}$$ $\tilde{M}$ larger, open manifold without boundary.

M has bdry.

$$\hat{H}^m(M) = \{u : D^\alpha u \in L^2,\ |\alpha| \le m\}.$$

Seeley: $\mathcal{E} : \hat{H}^m(M) \to H^m(\tilde{M})$.



$f \in C^\infty(\overline{\mathbb{R}^+})$

$\mathcal{E} : C^\infty(\overline{\mathbb{R}_+}) \to C^\infty(\mathbb{R})$,
$\mathcal{E}(f)|_{\overline{\mathbb{R}^+}} = f$.

$H^m_0(\Omega)$ = closure of $C_0^\infty(\mathring{M})$ in H^m norm.

Note: This is a proper, closed subset.

$$r_0 : H^m(M) \to H^{m-\frac{1}{2}}(\partial M).$$
$$r_j : H^m(M) \to H^{m-j-\frac{1}{2}}(\partial M).$$

So, $H_0^m(M) \subset \bigcap_{j=0}^{m-1} (\ker \gamma_j)$.

Weak formulation: $\Delta u = f$, $\gamma_0 u = h_0$.

$u \in H^1$, expect $f \in H^1(M)$.

If u, v smooth

$$\langle \nabla u, \nabla v \rangle = \langle \Delta u, v \rangle + \int_{\partial M} (\gamma_0 u)(\gamma_1 v).$$

So,

$$\langle \Delta u, \Delta u \rangle = \langle \Delta u, v \rangle + \int_{\partial M} u_0 v_1 - u_1 v_0.$$

$v \in C^\infty(\bar{M})$.

$$\langle u, \Delta v \rangle = \langle f, v \rangle + \int_{\partial M} (h_0 v_1 - u_1 v_0).$$

If $v \in H_0^1$ (or $v \in C^\infty(\bar{M})$, $\gamma = 0$) then

$$\langle u, \Delta v \rangle = -\langle f, v \rangle + \int_{\partial M} h_0 v_1.$$

Want remove this term —.

~~Mode~~ Now

Model Problem:

$$\sum D_{x_j}^2 + 1 = \Delta + 1.$$

$$(\Delta + 1) u = f \quad \text{on } \mathbb{R}^n_+.$$

$$u_0 = h \quad \text{on } \mathbb{R}^{n-1}.$$

$$x = (x_1, \dots, x_{n-1}, x_n) = (y, t).$$

$$(-\partial_t^2 + \Delta_y + 1) u = f.$$

$$(-\partial_t^2 + \underbrace{(1 + |\eta|^2)}_{e^{\pm\sqrt{1+|\eta|^2}\, t}}) \tilde{u}(t,\eta) = \tilde{f}(t,\eta).$$

$$u(t,y) = \int e^{iy\cdot\eta - t\sqrt{1+|\eta|^2}} A(\eta)\, d\eta.$$

$$u(0,y) = \int e^{iy\cdot\eta} A(\eta)\, d\eta \Rightarrow A = \tilde{u}(\eta).$$

$$u(t,y) = \int e^{i(y-\tilde{y})\cdot\eta - t\sqrt{1+|\eta|^2}} u(\tilde{y})\, d\tilde{y}\, d\eta.$$

$$= \int K(t,y,\tilde{y}) u(\tilde{y})\, d\tilde{y}, \quad K(t,z) = \int e^{iz\cdot\eta - t\sqrt{1+|\eta|^2}} d\eta.$$

1) $K(0,z) = (2\pi)^n \delta(z)$.
2) ~~tx~~ For $t > 0$, $z \mapsto K(t,z)$ is in $\mathcal{S}$.
3) $\int K(t,z)\, dz = (2\pi)^n \underbrace{e^{-t}}_{\text{because } \Delta+1}$.

$$\cancel{\tilde{K}(t,\eta) = \int e^{t\sqrt{1+|\eta|}} d\eta}.$$

$$\tilde{K}(t,\eta) = \int e^{-iz} e^{t\sqrt{1+|\eta|}}\, dz.$$

At $\eta = 0$, we have 3.

$\Delta u = f$ in $\mathbb{R}^n_+$, $u_0 = h$.

We essentially solved $\Delta w = f$, $w_0 = h$.

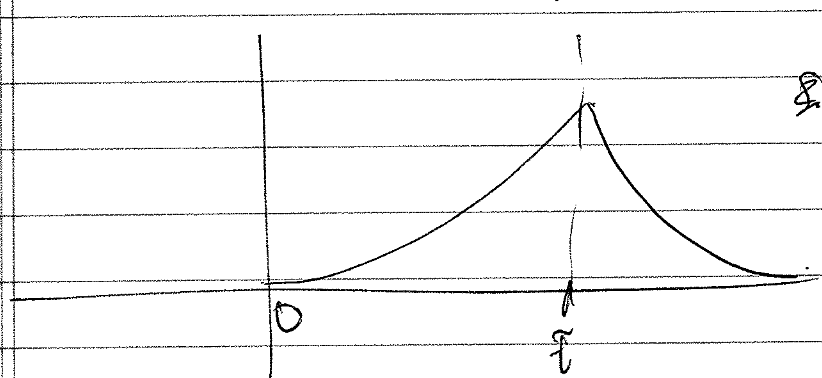
$$\left(-\partial_t^2 + (1+|\eta|^2)\right)\tilde{w} = 0, \quad \tilde{w}_0 = \hat{h}.$$
$$\tilde{w} = \hat{h}(\eta)\, e^{-t\sqrt{1+|\eta|^2}}.$$

To get to $\Delta v = f$, $v_0 = 0$, want Greens fun.

$$\tilde{v}(t,\eta) = \int_0^\infty \tilde{G}(t,\hat{t},\eta)\,\hat{f}(\hat{t},\eta)\,d\tilde{t}.$$

$$\left(-\partial_t^2 + (1+|\eta|^2)\right)\tilde{v} = \int_0^\infty \underbrace{\left(-\partial_t^2 + (1+|\eta|^2)\right)\tilde{G}(t,\hat{t},\eta)}_{\delta(t-\tilde{t})}\,\hat{f}(\tilde{t},\eta)\,d\tilde{t}.$$

$$\begin{cases} \left(-\partial_t^2 + 1 + |\eta|^2\right)\tilde{G}(t,\tilde{t},\eta) = \delta(t-\tilde{t}) = \begin{cases} 0 & t \neq \tilde{t} \\ \infty & t = \tilde{t} \end{cases} \\ \tilde{G}(0,\hat{t},\eta) = 0 \end{cases}$$



& Since we get δ $f^{\tilde{=}}$, two solutions on either side of $\tilde{t}$.

Two homogenous sol^a's. $e^{+t\sqrt{1+|\eta|^2}}$, $e^{-t\sqrt{1+|\eta|^2}}$.

$B_1(\tilde{t})\sinh(t\sqrt{1+|\eta|^2})$. $B_2(\tilde{t})e^{-t\sqrt{1+|\eta|^2}}$.

sinh comes in because we get 0 at $t=0$.
So,

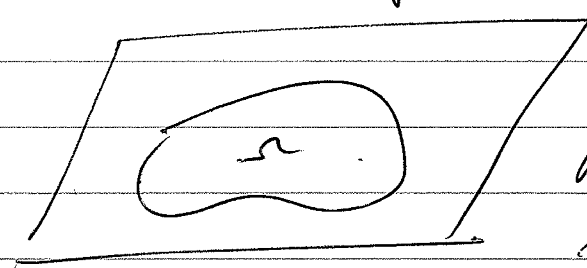
$$G(t,\tilde{t},\eta)=\begin{cases} \cancel{e}^{-\tilde{t}\sqrt{1+|\eta|^2}} \sinh(t\sqrt{1+|\eta|^2}). & t<\tilde{t}. \\ \sinh(\tilde{t}\sqrt{1+|\eta|^2})\, e^{-t\sqrt{1+|\eta|^2}}. & t>\tilde{t}. \end{cases}$$

<u>General Method:</u>

$$Pu=f$$
$$B(\gamma u)=h.$$

$M\subset M'$. closed mfld.

On $\mathbb{R}^n$,



put in box and identify sides to it it inside $\mathbb{T}$.

Extend P to P' on M'. ← lots of choices, but happy to be modulo C^∞.

Extend f to f' on M'.

Solve $P'u'=f'$ on M'.

<u>Problem.</u> u' is not going to solve right bdy conditions.

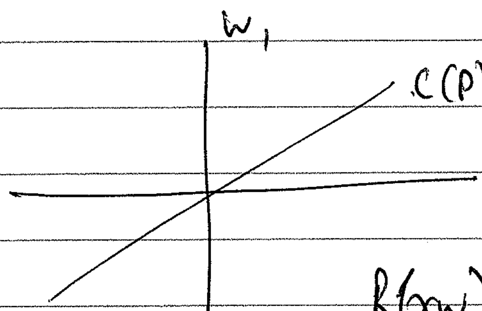
$v = u'|_M$, $Pv = f$ $(+ \partial f)$.

Problem: $B(rv) = k \neq h$.

$\boxed{\text{Solve } Pw = 0, \ B(rw) = h - k.}$ Still a bdy problem but easier.

<u>Calderon's Idea:</u>

$Pw = 0$, $B(rw) = \varphi$.

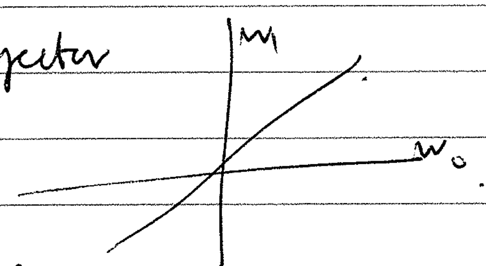


$C(P) = \{rw : Pw = 0\}$. Closed subspace in an appropriate topology.

$B(\gamma w) = (\sum B_{ij} w_j) = (\varphi_i)$.

$B \cdot (rw) \xrightarrow{\sim} \varphi$. (want this to be an isomorphism on $C(P)$).

$\mathcal{C}$ - Calderon projector



ran $\mathcal{C} = C(P)$. (project onto subspace).

$\mathcal{C} : \bigoplus_{k=0}^{m-1} H^{m-k-\frac{1}{2}} \circlearrowleft$

Fact (1) $\mathcal{C}$ is a matrix of pseudo-diff ops!
(2) $\sigma(B) = (\sigma(B_{ij}))$.

$\sigma(\mathcal{C}) : \mathbb{C}^m \to \mathbb{C}^m$. (matrix of the symbols of $\mathcal{C}$)

Ellipticity $\iff$ $\sigma(B) : \operatorname{ran}(\sigma(\mathcal{C})) \to \mathbb{C}^r$ $\boxed{\text{Isom}}$.

Take Δ. $\sigma(\mathcal{C}_\Delta) = \begin{pmatrix} \frac{1}{2} & \frac{i|\eta|^{-1}}{2} \\ \frac{-i|\eta|}{2} & \frac{1}{2} \end{pmatrix}$.

$\operatorname{ran} \sigma(\mathcal{C}_\Delta) = \{w_1 = |\eta| w_0\}$.