Lecture 7 (Copied from P. Hintz).

Def. $u \in \mathcal{D}'(\Omega) \to WF(u) = \bigcap \{ \mathrm{Char}(P) : P \in \Psi^0, Pu \in C^\infty \}$

Lemma. $u \in \mathcal{E}'$ (compactly supported distrs?), $\varphi \in C_0^\infty$.

$\Rightarrow WF(\varphi u) \subset WF(u)$.

Fact: $u, v \in \mathcal{E}'$, $WF(u) + WF(v) \not\ni 0$

$\Rightarrow uv \in \mathcal{E}'$ exists and extends mult of smooth functions.

Pf of Lemma. Let $w = \varphi u \in \mathcal{E}'$

$$\Rightarrow \hat{w}(\xi) = \int \hat{\varphi}(\eta)\, \hat{u}(\xi - \eta)\, d\eta.$$

Have $|\hat{u}(\xi)| \le C(1+|\xi|)^{-m}$ (some m, C, all ξ).

Choose $0 < c < 1$.

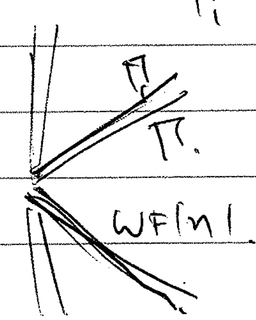
On $|\eta| \ge c|\xi|$, $|\xi - \eta| \le (1 + c^{-1})|\eta|$.

$|\eta| \le c|\xi|$, $|\xi - \eta| \le (1+c)|\xi|$.

$$\Rightarrow |\hat{w}(\xi)| \lesssim \sup_{|\xi-\eta| \le (1+c)|\xi|} |\hat{u}(\eta)| \cdot \|\hat{\varphi}\|_{L^1} + \int_{|\eta| \ge c|\xi|} |\hat{\varphi}(\eta)|(1+|\eta|)^m\, d\eta.$$

$\cancel{\|\hat{\varphi}\|_{L^1}}$

$$\Rightarrow \sup_{\Gamma_1} (1+|\xi|)^N |\hat{w}(\xi)| \le C \sup_{\Gamma} |\hat{u}(\eta)|(1+|\eta|)^N \|\hat{\varphi}\|_{L^1} + \int_{|\eta| \ge c|\xi|} |\hat{\varphi}(\eta)|(1+|\eta|)^{N+m}\, d\eta < \infty$$



∎

Prop. $\pi\, WF(u) = \text{sing spt } u$, where $\pi(x,\xi) = x$.

Pf. • $x_0 \notin \text{sing spt } u \Rightarrow \chi \in C_0^\infty$, $\chi(x_0) = 1$ s.t. $\chi u \in C_0^\infty$, then $P = \chi \in \Psi^0$.
Char(P) disjoint from $\{(x_0,\xi) : \xi \in \mathbb{R}^n\}$
$\Rightarrow (x_0,\xi) \notin WF(u) \quad \forall \xi$.

• $x_0 \notin \pi\, WF(u)$, ie $\forall \xi \ (x_0,\xi) \notin \pi\, WF(u)$.
$\Rightarrow \exists Q_\xi \in \Psi^0$, $(x_0,\xi) \notin \text{Char}(Q_\xi)$, $Q_\xi u \in C^\infty$.

Choose $Q_1, \dots, Q_N$ s.t. $\forall \xi\ \exists j$ s.t. $\xi \notin \text{Char}(Q_j)$.
Put $Q = \sum_{j=1}^N Q_j^* Q_j \in \Psi^0$ $\quad \sigma_0(Q) = \sum |\sigma_0(Q_j)|^2$
on $\{(x_0,\xi) : |\xi| = 1\}$.
$\Rightarrow Q$ elliptic near x_0 and Qu is smooth.
Apply local paramatrix. $\square$

Def. Ess sup P for $P = Op(p(x,\xi))$ smallest closed set s.t. $P \sim 0$ on the compliment.

Note: $\text{Ess spt}(P_1 P_2) \subset \text{Ess spt}(P_1) \cap \text{Ess spt}(P_2)$.
(use $\sigma(P_1 P_2) = \sum \frac{i^{|\alpha|}}{|\alpha|!} D_\xi^\alpha p_1\, D_x^\alpha p_2$).

Lemma. $u \in \mathcal{E}'$, $\mathcal{U}$ open conic, $WF(u) \cap \mathcal{U} = \emptyset$.
If $\text{Ess spt}(P) \subset \mathcal{U}$, then $Pu \in C^\infty$.

Pf. Assume $P \in \Psi^0$. Can find Q_i s.t. $Q_i u \in C^\infty$
and every $(x,\xi) \in \text{ess spt}(P)$ is non-characteristic for some

Q_j, $j = 1, \dots n$. Set $Q = \sum Q_j^* Q_j$. Then $Qu \in C^\infty$ and $\mathrm{char}(P) \cap \mathrm{ess\,supp}(P) = \emptyset$, $\exists$ local parametrix $B \in \Psi^0$ $\mathrm{ess\,supp}(B \circ Q - I) \cap \mathrm{ess\,supp}(P) = \emptyset$.

$\Rightarrow$ with A s.t. $A \circ Q \sim P$ in Γ. (take $A = P \circ B$).

$Pu = AQu \in C^\infty$. ∎

Prop $WF(Pu) \subset WF(u) \cap \mathrm{Ess\,supp}(P)$.

Cor P elliptic $\Rightarrow$ $WF(Pu) = WF(u)$.

Now Elliptic boundary problems.

M smooth mfld w/ bdry, or $M = \Omega \subset \mathbb{R}^n$ w/ $\partial\Omega$ smooth.

Consider P = ell. diff. op. in Ω of order m.

For u = fcn, eg $u \in H^m(\mathbb{R}^n)$, let $\mathrm{tr}\, u = \begin{pmatrix} u_0 \\ \vdots \\ u_{m-1} \end{pmatrix}$,

$$u_j = \left(\frac{\partial_\nu}{i}\right)^j u \Big|_{\partial\Omega}.$$

Like to solve
$$\begin{cases} Pu = f & \text{in } \Omega. \\ B\begin{pmatrix} u_0 \\ \vdots \\ u_{m-1} \end{pmatrix} = h, \end{cases}$$

?

where $B(\mathrm{tr}\, u) = \left(\sum_{k=0}^{m-1} B_{jk} u_k\right)_{j=1,\dots,l}$, $B_{jk} \in \Psi^{-}(\partial\Omega)$

Eg. $\begin{cases} \Delta u = f \\ u_0 = h \end{cases}$ (Dirichlet).

$\begin{cases} \Delta u = f \\ u_1 = h \end{cases}$ (Neumann)

$\begin{cases} \Delta u = f \\ B_0 u_0 + B_1 u_1 = h \end{cases}$ (Generalized Robin condition).

<u>Q.</u> when does this problem have good properties?
(i.e. solvability, regularity).

<u>Want</u> $\forall f \in L^2$; $h_j \in H^{\sigma_j}$ ($B_{ju}: H^{m-k-\frac{1}{2}} \to H^{\sigma_j}$)
$j = 1, \dots, l$, $\exists! u \in H^m(\Omega)$. (up to finite dim ker, coker).

<u>Model problem.</u> $P = -\Delta + 1$ on $\mathbb{R}^n_+$,
$P = \sum_{j=1}^n D^2_{x_j} + 1$.

FT in $y \rightsquigarrow$ op has $-\partial^2_{x_n} + (|\eta|^2 + 1)$.

Io, $Pu = f(x_n, y) \to (-\partial^2_{x_n} + (1 + |\eta|^2)$
$\tilde{u}(x_n, \eta) = \tilde{f}(x_n, \eta)$.

Need hom. sol^{n}s: $(-\partial^2_{x_n} + (1 + |\eta|^2))\tilde{u} = 0$.
$\to \sum_{\pm} e^{\pm(1+|\eta|^2)^{\frac{1}{2}} x_n} a_{\pm}(\eta)$.

$(+)$ - term is bad - exp growth.

$$\Rightarrow \tilde{u}(x_n, \eta) = A(\eta)\, e^{-(1+|\eta|^2)^{\frac{1}{2}} x_n}.$$

Hence $Pu = 0$, can only prescribe A!

Eg $\begin{cases} Pu = 0 \\ u(0,y) = h \end{cases} \longrightarrow$ $\tilde{u}(x_0, \eta) = A(\eta)\, e^{-\sqrt{1+|\eta|^2}\, x_n}$

$x_n = 0 \rightsquigarrow A(\eta) = \tilde{h}(\eta)$.

$\tilde{u}(0, \eta) = \tilde{h}(\eta)$. invertible condition.

or $\partial_{x_n}(0, y) = h_1 \Rightarrow -\sqrt{1-|\eta|^2}\, A(\eta) = \tilde{h}_1(\eta)$.

<u>Cannot have both at once!</u>