Lecture 6 (Copied from Peter Hintz).

$A \in \Psi^m(M)$, M cpct. A elliptic $\Rightarrow$ $A : H^{s+m}(M) \to H^s(M)$.
$\Leftarrow$ Fredholm.

Def. A right elliptic if $\sigma_m(x,\xi)$ is injective.
left. if $\sigma_m(x,\xi)$ is surjective.

Ex. $\operatorname{div} : C^\infty(M, TM) \to C^\infty(M)$.

$$X \longmapsto \sum_j \langle \nabla_{e_j} X, e_j \rangle .$$

$$\nabla_{e_j} X = \left(\sum_i \left(\frac{\partial X_i}{\partial x_j} + \Gamma^k_{ij} \right) \partial_{x_k} \otimes dx^j \right)(e_i)$$

$$\sigma_1(\operatorname{div})(x,\xi) = \sum \xi^i X_i = \xi(X).$$

$\to$ not elliptic, not right elliptic, but left elliptic.

Ex. $d : C^\infty(M) \to C^\infty(M, T^*M)$. $f \mapsto \sum \frac{\partial f}{\partial x_j} dx^j$.

$$\sigma_1(d)(x,\xi) : f \mapsto f(\xi_j\, dx^j) = f\xi, \text{ right elliptic.}$$

Thm If A is left or right elliptic, then
$A : H^{s+m} \to H^s$ has closed range.

Generally, A left ell. $\Rightarrow$ $\operatorname{coker} A$ inf-dim.
right ell $\Rightarrow$ $\ker A$ inf-dim.

Eg. $\{ w \in L^2(M, T^*M) : w = df,\ f \in H^1(M) \} \subset L^2(M, T^*M)$ closed subspace

Pf. Sps A right elliptic. $\Rightarrow A^*A \in \Psi^{2m}$ has symbol $\sigma_{2m}(A^*A) = \sigma_m(A^*)^* \sigma_m(A)$, which is injective, hence bijective. Choose $B \in \Psi^{-m}$ s.t. $BA^*A \sim I$ and define $G = BA^* \in \Psi^{-m}$. Then, $GA = I - Q$. Now, if $Au_j = f_j$ is ~~the~~ Cauchy.
$\Rightarrow G(Au_j) = Gf_j = u_j - Qu_j$, ie $u_j = \underbrace{Gf_j}_{\text{Cauchy}} + Qu_j$

Write $u_j = v_j + w_j$, $v_j \perp \ker A$, $w_j \in \ker A$.

$\Rightarrow Au_j = Av_j = f_j \rightsquigarrow$ replace u_j by v_j.

If v_j bdd $\rightarrow Qv_j$ has convergent subsequence.
Else $A\left(\frac{v_j}{\|v_j\|}\right) = \frac{f_j}{\|v_j\|} \to 0$ hence $\underbrace{\text{limit}}_{\text{compactness of } Q}$ of $\frac{v_j}{\|v_j\|}$ is in $\ker A$.

But $\perp \ker A$ and norm $\neq 0$ so contradiction. $\boxtimes$.

Hence. A right elliptic $\Rightarrow$ ran A closed, $\ker A$ finite dim. ($2/3$ Fredholm).

Prop. A has closed range $\Leftrightarrow A^*$ has closed range.

Prop. $\operatorname{div} = \delta : H^1(M, TM) \to L^2(M)$.
$\rightsquigarrow$ ~~$\forall$~~ $X \in H^1(M, TM)$ $X = X_0 + X_1$ with.

$X_0 \in \ker(\operatorname{div})$, $\delta X_0 = 0$, $X_1 = \nabla f$.

$WF(u)$: wave front set of $u \subset T^*X \setminus \{0\}$.

Basic fact: $u \in C_0^\infty(\Omega) \Rightarrow \hat{u}(\xi) \in \mathcal{S}$, i.e. satisfies
$$|\hat{u}(\xi)| \lesssim (1+|\xi|)^{-N} \quad \forall N.$$

More generally, $\mathbb{R}^n$ say $u \in \mathcal{S}$ in $\Gamma \subset \mathbb{R}^n$ open cone if $|\hat{u}(\xi)| \leq C_{N,\Gamma'}(1+|\xi|)^{-N} \quad \forall N, \forall \xi \in \Gamma' \subset \Gamma$.

Look at, e.g., $\varphi(x) H(x_1)$ where $H(x_1) = \mathbb{1}_{[0,\infty)}(x_1)$ $\varphi \in C_0^\infty(\mathbb{R}^n)$.
(H_1 just does a cutoff to positive plane).

- First $H' = \delta$ (in $n=1$) $\Rightarrow \xi \hat{H} = 1 \Rightarrow$ "$\hat{H} = \frac{1}{\xi}$".
(problem: homogeneity $= -\dim \rightsquigarrow$ many possibilities to regularize).

- Next. $\varphi(x_1) H_1(x_1)$ has F-T. $\hat{\varphi}(\xi) * \hat{H}(\xi) \sim \frac{1}{\xi}$.
- Lastly, $\varphi_1(x_1) H_1(x_1) \varphi_2(x_2) \cdots \varphi_n(x_n)$
$$\rightsquigarrow |FT| = |\widehat{\varphi_1 H}(\xi_1) \hat{\varphi}_2(\xi_2) \cdots \hat{\varphi}_n(\xi_n)|$$
$$\lesssim (1+|\xi_1|)^{-1}(1+|\xi'|)^{-N}.$$

which is Schwartz in open cone around $(\xi_1, \dots, \xi_n)$ iff $(\xi_2, \dots, \xi_n) \neq 0 \in \mathbb{R}^{n-1}$.

<u>Def^n</u> $(x_0, \xi_0) \notin WF(u)$ if
- $\exists \chi \in C_0^\infty(U)$, $\chi(x_0) \neq 0$.
- $\exists \Gamma \subset \mathbb{R}^n$ open cone, $\xi_0 \in \Gamma$

s.t. $(\chi(x) u(x))^\wedge(\xi)$ is rapidly decaying in Γ.

Equiv. $WF(u) = \bigcap_{\chi, \Gamma} \{(x,\xi) : \chi(x) \neq 0, \xi \in \Gamma, (\chi u)^\wedge$

~~not~~ not rapidly decaying in cone Γ near $x\}$.

My note. You are not in the WF set if you can find a neighborhood ~~and cone inside which~~ cutoff χ inside which and a cone inside which, s.t. $|\chi u|^\wedge$ is rapidly decaying.

Thm. 1) $A \in \Psi^m$, $u \in \mathcal{D}' \Rightarrow WF(Au) \subset WF(u)$.

2) $WF(u) \subset WF(Au) \cup \underbrace{\mathrm{Char}(A)}_{\text{later}}$.

3) A ell $\Rightarrow WF(Au) = WF(u)$.

Lemma. With $\pi : T^*M \to M$, $\pi WF(u) = \text{sing spt } u$.

Pf. • $x_0 \notin \text{sing spt } u \Rightarrow \exists \chi \in C_0^\infty$, $\chi(x_0) \neq 0$ s.t. $\chi u \in C_0^\infty \Rightarrow \widehat{\chi u}$ rapidly decaying in all directions.

• $x_0 \notin \pi WF(u) \Rightarrow$ For any ξ, $|\xi| = 1$, $(x_0, \xi) \notin WF(u)$

$\Rightarrow \exists \varphi_\xi$, $\varphi_\xi(x_0) \neq 0$, s.t. $\widehat{\varphi_\xi u}$ is rapidly decaying in some open cone Γ_ξ.

Choose $\xi_1, \dots, \xi_N \in \mathbb{R}^n \setminus \{0\}$. s.t. $\Gamma_{\xi_1} \cup \dots \cup \Gamma_{\xi_N} = \mathbb{R}^n \setminus \{0\}$

$\varphi(x) = \varphi_{\xi_1}(x) \cdots \varphi_{\xi_N}(x) \Rightarrow \widehat{\varphi u}$ rapidly decays.

$\Rightarrow u$ smooth near x_0. $\square$

Pf. of thm 1). $Au(x) = \int e^{i(x-y)\xi} a(x,\xi) u(y)\, dy\, d\xi$.

$\Rightarrow \left(\chi(x) Au(x)\right)^\wedge(\xi) = \int \chi(x) e^{ix\eta - ix\xi - iy\eta} a(x,\eta) u(y)\, dy\, dx\, d\eta$

$= \int \hat{\chi a}(\xi-\eta, \eta) \cdots u(y) e^{-iy\eta} dy\, d\eta$.

$= \int \widehat{\chi a}(\xi-\eta, \eta)\, \hat{u}(\eta)\, d\eta$ — finish up.

2). $\mathrm{Char}\, A = \{(x,\xi) \neq \ \sigma_m(A)(x,\xi) = 0\}$.

Prop. Choose any $A = op(a) \in \Psi^m$. Suppose that in ~~supp~~ $\mathrm{spt}_x\, a(x,\xi)$, $\exists\ \Gamma \subset \mathbb{R}^n$ open cone s.t. $|a(x,\xi)| \geq c(1+|\xi|)^m$ $\forall \xi \in \Gamma' \subset \Gamma$ here.
Then, $\exists\ B \in \Psi^{-m}$ s.t. $BA - I = Q$ satisfies $\sigma(Q) \in S^{-\infty}(\Gamma')$.

Pf. obvious, as for elliptic operators.

Example. $A = D_t^2 - \Delta$, $\sigma_2(A) = \tau^2 - |\xi|^2$ prop. applicable ~~in cones~~. ~~Outside of light cone~~ in cones outside of light cone. Use of this? Sps $Au = f \in C_0^\infty$ where $u \in \mathcal{D}'$.
$\Rightarrow B(Au) = u - Qu = Bf \Rightarrow u = Bf + Qu$ does not have WF in such cone.
$\Rightarrow WF(u) \subset \mathrm{Char}\, A$ "if Au is smooth.

Pf. of thm 3 is just this!. $\blacksquare$