Rafe Mazzeo Lec 5 — 03/10/2012.

Last time: $\Omega \subset \mathbb{R}^n$ open.

$a(x,\xi) \in S^m(\Omega \times \mathbb{R}^n)$, elliptic.

$A = op(a)$.

modify by $R \in \Psi^{-\infty}$ so that $A = op(a) + R$ is properly supported.

Don't need to worry about this if $op(a)$ is a diff. operator since diff operators are supported on the diagonal.

Constructed B, properly supported s.t.

$B \circ A = I - Q_1, \quad A \circ B = I - Q_2, \quad Q_1, Q_2 \in \Psi^{-\infty}$.

First consequence: local elliptic regularity $Au = f \in \mathcal{D}'(\Omega)$

but suppose $f|_{\mathcal{U}} \in H^s(\mathcal{U})$ i.e., $\langle f, \varphi \rangle = \int h_{\mathcal{U}} \varphi$

$\forall \varphi \in C_0^\infty(\Omega)$. ~~Then $u \in H^{s+m}$~~ in $\mathcal{U} \subset\subset \Omega$.

Then, $u \in H^{s+m}(\mathcal{U})$.

Pf. Take $\chi \in C_0^\infty(\mathcal{U})$. Then,

$$u|_{\mathcal{U}} \in H^{s+m}(\mathcal{U}) \Leftrightarrow \chi u \in H^{s+m}, \ \forall \chi \in C_0^\infty(\mathcal{U}).$$

$$Au = f, \quad f = \chi f + (1-\chi f), \quad \chi \equiv 1 \text{ on } \mathcal{U}'' \subset\subset \mathcal{U} \subset \Omega.$$

$$B \circ A u = u - Qu = \underbrace{B(\chi f)}_{\uparrow\ H^{s+m}} + \underbrace{B((1-\chi) f)}_{C^\infty(\mathcal{U}'') \text{ since pseudo locality. No singularities in } \mathcal{U}''}$$

Like u to be cptly spted. So, write.

$$\chi A u = \chi f, \text{ but } \chi A \text{ is no longer elliptic.}$$

Write $\chi A = A\chi + [\chi, A]$

$$\sigma_m(\chi A) = \sigma_m(A\chi) + \cancel{\sigma_m([\chi, A])}\ 0$$

$$\Rightarrow \chi A = A\chi + E, \quad E \in \Psi^{m-1}.$$

Supps $u|_{\bar{\mathcal{U}}} \in H^t$ (every distribution lives in Sobolev space in fixed cpt set.)

$$\Rightarrow Eu \in H^{t-m+1}$$

$$(B \circ A)(\chi u) = \chi u - Q(\chi u) = -B \circ Eu + B(\chi f).$$

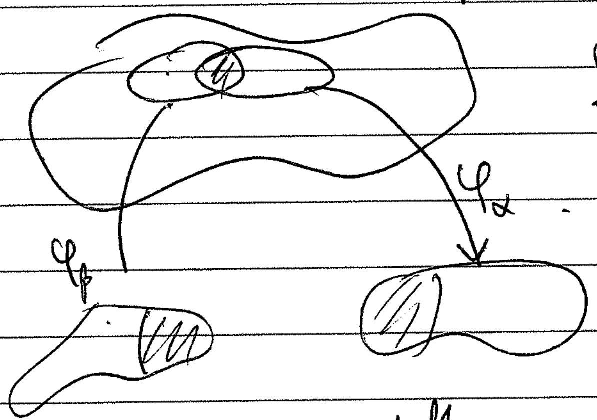
$$\text{So, } \chi u = Q_1(\chi u) + (B \circ E) u + B(\chi f) \in C^\infty + H^{t+1} + H^{s+m}.$$

If we iterate this enough, we can get the $t+1 > H^{s+m}$ and then we stop.

Today: Localising & Microlocalising.

"Going Up"

M.



$\{U_\alpha\}$.

diffeo.

$F: \Omega \to \Omega'$; $v \in \mathcal{D}'(\Omega')$. Ω' domain in ~~M~~ $\mathbb{R}^n$.

$$\langle F^* v, \varphi \rangle \overset{"}{=} \int v(F(x))\,\varphi(x)\,dx.$$

$$= \int v(y)\,\varphi(F^{-1}(y))\,(\det F^{-1}_*)\,dy \text{ "}.$$

$$= \langle v, (F^{-1})^* \varphi\,(\det F^{-1}_*) \rangle.$$

$A(x, D)$ diff op $\Omega \subset \mathbb{R}^n$.

$$A_F v = (F^{-1})^* A F^* v, \quad v \in C^\infty(\Omega').$$

$$\sum a_\alpha(x) D^\alpha_x (v(F(x))) \Big|_{x = F^{-1}(y)}.$$

$$= \sum b_\beta(x) (D^\beta_y v) \Big|_{y = F(x)}.$$

$$Au(x) = \int e^{i(x-y)\xi} a(x,\xi)\, u(y)\, dy\, d\xi .$$

$$A(F^* v) = \int e^{i(x-y)\xi} a(x,\xi)\ a(x,\xi)\ v(F(y))\ dy\, d\xi .$$

$$(F^{-1})^* A(F^* v) = \int e^{i(F^{-1}(x') - F^{-1}(y'))\xi} a(F^{-1}(x'), \xi) v(y')\ JF\ dy' d\xi$$

But this doesn't look like pseudo. diff.

$$F^{-1}(x') - F^{-1}(y') = C(x',y')(x'-y') \qquad C \text{ matrix}$$

So, $$= \int e^{i(x'-y')\cdot\eta} \cdot a(F^{-1}(x'), (C(x',y')^t)^{-1}\eta)\, \tilde{J}\, v(y)\, dy\, d\eta .$$

$$\eta = C(x',y')^t \xi .$$

$\boxed{\tilde{J} \text{ only depends on } (x'),(y').}$

__Upshot:__ $$(A_F v)(x') = \int e^{i(x'-y')} \tilde{a}(x',y',\eta) v(y')\, dy'\, d\eta .$$

__Note:__ $\tilde{a}$ absorbs the Jacobian, which is just a smooth function!

__Claim:__ $a \in S^m \Rightarrow \tilde{a} \in S^m$. a elliptic $\Rightarrow \tilde{a}$ elliptic.

__Key fact:__ $\sigma_m(a) \in S^m$ is well defined on T^*M modulo S^{m-1}!

This is because the change is exactly by $C(x',y')^t$ which is the way in which covectors change.

Th^{m}. $A \in \Psi^m(M)$ M cpct, A elliptic. Then,
$A : H^{s+m}(M) \to H^s(M)$ Fredholm for any $s \in \mathbb{R}$.
$\operatorname{Ker} A \subset C^\infty$, $(\operatorname{Rang} A)^\perp \subset C^\infty$.

All of this information is captured by the short exact sequence:

Global [illegible] in mflds.

$$\begin{cases} 0 \to \Psi^{k-1} \hookrightarrow \Psi^k(M) \to S^k/S^{k-1} \to 0. \\ \sigma_k(A) \in S^k(M, T^*M)/S^{k-1}(M, T^*M). \\ \sigma_{k_1}(A_1) \circ \sigma_{k_2}(A_2) = \sigma_{k_1+k_2}(A_1 \circ A_2) \mod S^{k_1+k_2-1}. \\ \sigma_k(A^*) = \sigma_k(A)^* \mod S^{k-1}. \end{cases}$$

Pf. ~~A~~ (of Th^{m}). $A \in \Psi^m(M)$ elliptic.

Find $B_{-m} \in \Psi^{-m}$ s.t. $A \circ B_{-m} - I \in \Psi^{-1}$.

$$\sigma_m(B_{-m}) = \sigma_m(A)^{-1} \mod S^{m-1}.$$

Continue: B_{-m-1} s.t. $A\circ(B_{-m}+B_{-m-1})-I \in \Psi^{-2}$.

Hence, $A: H^{s+m} \to H^s$ bdd.

① A is Fredholm: $A\circ B = I$, $B \circ A = I$ mod $\Psi^{-\infty}$.

$u \in \ker A \Rightarrow Au = 0 \Rightarrow B(Au) = u - Q_1 u = 0$.
$\Rightarrow u = Q_1 u \in C^\infty(M)$.

Thus, $Id|_{\ker A} = Q_1|_{\ker A}$

is compact $\Rightarrow \ker A$ finite dim.

$(\operatorname{ran} A)^\perp \cong \ker A^*$ finite dim by same argument

$Au_j = f_j \xrightarrow{H^s} f \Rightarrow u_j = Bf_j + Qu_j$

$Bf_j \to Bf$ in H^{s+m}.

- $\|u_j\|_{H^{s+m}} \le C \Rightarrow Q_j$ converges.

- If not $u_j \perp \ker A$, $Au_j = f_j$ and
dps $\|u_j\|_{H^{s+m}} = C_j \to \infty$.

$A\left(\dfrac{u_j}{C_j}\right) = \dfrac{f_j}{C_j} \to 0 \Rightarrow \dfrac{u_j}{C_j} \to \bar{u}$ in H^{s+m}

$A\bar{u} = 0$ by above.

But this contradicts $\dfrac{u_j}{C_j} \perp \ker A$. ∎

If A elliptic $\Rightarrow$ Fredholm.

$$\mathrm{ind}(A) = \dim \ker A - \dim \operatorname{coker} A.$$

Th$^{\underline{m}}$ K compact. $K: H^{s+m} \to H^s$, then.

$$\mathrm{ind}(A+K) = \mathrm{ind}(A).$$

Atiyah - Singer Index Theorem.

$$\mathrm{Ind}(A) = \int_M (AS)(\sigma_m(A))$$

$\nearrow$ formula. $\uparrow$ only depends on symbol!

(Actually implies the Gauss-Bonnet th$^{\underline{m}}$!).

(M^n, g). $\Omega^0 \xrightarrow{d} \Omega^1(M) \to \Omega^2(M) \to \cdots \to \Omega^n(M) \to 0$.

Ω^k = diff k-forms.

$$\omega = \omega_I \, dx^I, \quad d\omega = \frac{\partial \omega_I}{\partial x_j} dx^j \wedge dx^I \in \Omega^{k+1}.$$

$$d \circ d = 0.$$

$$\langle d\omega, \eta \rangle = \langle \omega, \delta\eta \rangle, \quad \delta = d^*.$$

$$D = d + \delta : \Omega^{\text{even}} \to \Omega^{\text{odd}}.$$

Claim. D is elliptic.

$D^2 = (d+\delta)(\delta+\delta) = d\delta + \delta d$ preserves degree.

$\sigma_2(D^2) = |\xi|^2 \cdot Id_{\Lambda^k}$

$\Rightarrow \sigma_1(D) = |\xi|^2 \Rightarrow \sigma_1(D)$ injective.

Also surjective by examining rank.

$\omega \in \ker D \Rightarrow D\omega = 0$, $\omega = \sum_{k=0}^{n/2} \omega_{2k}$.

$\Rightarrow D^2\omega = 0 = \Delta\omega \Rightarrow \Delta\omega_{2k} = 0$.

$\dim \ker(D) = \oplus\, b_{2k}(M)$ $\quad b_{2k}$ Betti numbers.

$\dim \operatorname{coker} D = \oplus\, b_{2k+1}(M)$.

$\operatorname{Ind}(D) =$ ~~Euler~~ $\chi(M)$, Euler characteristic

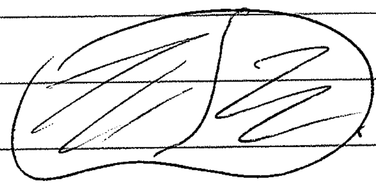
$\chi(M) = \int_M k \cdot dA \quad \dim M = 2$.

$= \int \underbrace{Pff(R_M)}$. $\quad \dim =$ even.

complicated but natural object on curvature.

$\boxed{\text{Patodi '68}}$ — computed a different ~~eq~~ integrand using parametrics.

<u>Next time</u>: $u \in \mathcal{D}'(\Omega)$, sing spt u — smallest closed set where u is not smooth.



C for scalar pseudo-diff.

A elliptic $\Rightarrow$ sing spt$(Au) \overset{\downarrow}{=}$ sing spt(u).

$u \in \mathcal{D}'(\Omega)$, $WF(u) = \{(x,\xi) : \sim\}$

$\downarrow$ proj

sing spt u.

So, WF(u) wavefront set of u is ~~exact~~ a refinement, ~~If you~~

- If you travel along singular set, won't see it. Only when you are travelling transverse. Thus, WF(u) measures position <u>and</u> direction.