~~Ra~~ Rafe Mazzeo PDE Lecture 2.
(Notes from notes).

**Def$^{\underline{n}}$** $P(x,D) = \sum_{|\alpha|\le m} a_\alpha(x) D^\alpha$ is called *elliptic* if

$$\sum_{\alpha} a_\alpha(x)\xi^\alpha = P_m(x,\xi) \text{ is invertible for } \xi \neq 0.$$

$$\left( \Rightarrow \frac{1}{P_m(x,\xi)} \le C(1+|\xi|)^{-m} \right)$$

Here $\sigma_m(P)(x,\xi) = P_m(x,\xi)$ is the *principal symbol* of $P$, is an invariant object $\in C^\infty(T^*\Omega)$, $\Omega \subset \mathbb{R}^n$ open.

(I) Explicit computation: $x \mapsto y(x)$ gives $D_{x_j} = \frac{\partial y_k}{\partial x_j} D_{y_k}$, and from this invariance can be deduced.

(II) coordinate-free formulation:

$$e^{-i\lambda\varphi(x)} P(x,D)\, e^{i\lambda\varphi(x)} = \sum_{|\alpha|\le m} P_\alpha(x)\, D_x^\alpha\, e^{i\lambda\varphi(x)}$$

$$= \sum P_\alpha(x)\, \lambda^m \left(\frac{\partial\varphi}{\partial x_1}\right)^{\alpha_1} \cdots + O(\lambda^{m-1}).$$

Then $\boxed{\sigma_m(P)(x_0, d\varphi(x_0)) = \lim_{\lambda\to 0} \lambda^{-m} e^{-i\lambda\varphi(x)} P(x,D)\, e^{i\lambda\varphi(x)}\Big|_{x=x_0}}$

**Goal.** Find a good class of $a$'s $(a = a(x,y,\xi))$, $x,y \in \Omega$, $\xi \in \mathbb{R}^n$, s.t.

$$Op(a)u(x) = \int e^{i(x-y)\xi} a(x,y,\xi)\, u(y)\, dy\, d\xi.$$

(oscillating integral).

has nice properties:

(I) invariant under composition,

(II) hold on nice functions.

(III) "inverses" of elliptic operators are of this sort

Will get for elliptic $P$ of order $m$

$\forall u \in C^\infty(M)$ $\|u\|_{H^m} \leq C(\|Pu\|_{L^2} + \|u\|_{L^2})$.

(M closed, otherwise $C^\infty(M) \cap L^2$).

(Note: this is just coercivity).

This gives:

(I) $\{u \in L^2 : Pu \in L^2\} = H^m$. Rellich: $H^m \hookrightarrow L^2$ cpct.

(II) $P : H^m \to L^2$ is Fredholm (or $P : L^2 \to L^2$ unbdd Fredholm)

- $\ker P$ finite dim
- $\operatorname{coker} P$ finite dim.
- $P$ is closed.

$\Rightarrow \exists G : \operatorname{ran}(P) \to (\ker P)^\perp$ equal to $P^{-1}$ there.

$G$ well defined anyhow. Extend by $0$ on $(\operatorname{ran} P)^\perp$ $\rightarrow$ $G \circ P = I - \pi_{\ker P}$ $P \circ G = I - \pi_{(\operatorname{ran} P)^\perp}$.

But now, spse $Pu = f$ and we have regularity info on $f$. So far, $G$ gives nothing.

Any $A : C^\infty_c(\Omega) \to \mathcal{D}'(\Omega)$, linear, cts

$\Rightarrow \exists K_A \in \mathcal{D}'(\Omega \times \Omega)$ s.t.

$\langle Au, v\rangle = \langle K_A, v(x)u(y)\rangle$ $\forall u, v \in C^\infty_c(\Omega)$.

$\rightsquigarrow$ should think of $G$ as

$$Gf(x) = \int G(x,y) f(y)\, dy.$$

Eg. On $\mathbb{R}^3$, $L = \sum D^2_{x_i} + 1$, $\sigma(L) = |\xi|^2$.

$$\rightsquigarrow G(x,y) = \frac{e^{-|x-y|}}{|x-y|} \quad \text{(mod constants)}.$$

Def$^n$. $a \in C^\infty(\Omega \times \Omega \times \mathbb{R})$ is called a symbol of order $m \in \mathbb{R}$ $\iff$ $a \in S^m$, I.e.,

$$\forall K \subset\subset \Omega \quad |D_x^\alpha D_y^\beta D_\xi^\gamma a(x,y,\xi)| \le C_{K,\alpha,\beta,\gamma} (1+|\xi|)^{m-|\gamma|}$$

Ex. Polynomials. $(1+|\xi|^2)^{\frac{1}{2}}$.

Now $P$ elliptic symbol $\rightsquigarrow$ $\dfrac{\chi(\xi)}{P(x,\xi)}$ $\left( \chi(\xi) = \begin{cases} 0 & |\xi| \le R \\ 1 & |\xi| \ge R+1. \end{cases}\right.$

$$\in S^{-m}$$

Special Case: classical (or polyhomogeneous) symbol.

$a \in S^m_{phg}$ (or $S^m_{cl}$) means

$$a \sim \sum_{j=0}^{\infty} a_{m-j}(x,\xi) \text{ where } \cancel{a_{m-j}(x,t\xi) = t^{m-j}}$$

$$a_{m-j}(x, t\xi) = t^{m-j} a_{m-j}(x, |\xi|) \quad |\xi| \ge 1.$$

Eg $\cdot$ $(1+|\xi|^2)^{\frac{1}{2}}$. means $\left( a - \sum_{j \le N} a_{m-j} \in S^{m-N-1} \right)$.

$\cdot$ $\chi(\xi) / p(x,\xi)$

Sum. $a \in S^m \ (m \in \mathbb{R}) \to \int e^{i(x-y)\xi} a(x,y)\xi \, d\xi. \in \mathcal{D}'(\Omega\times\Omega)$
$\cap \, C^\infty(\Omega\times\Omega \setminus \Delta)$
diagonal

more generally, $\Omega_x \times \mathbb{R}^N_\theta$, $\varphi(x,\theta)$ non-deg. phase function

(I) $\varphi(x, t\theta) = t\varphi(x,\theta)$, $t \geq 1$
(II) real valued
(III) $d_{(x,\theta)} \varphi \neq 0$. $(\theta \neq 0)$.

$$a(x,\theta) \in S^m(\Omega \times \mathbb{R}^N) \rightsquigarrow \int e^{i\varphi(x,\theta)} a(x,\theta) \, d\theta.$$
as distribution.

$$u \longmapsto I_{\varphi,a}(u) = \int e^{i\varphi(x,\theta)} a(x,\theta) u(x) \, d\theta \ dx.$$
$(u \in C_0^\infty(\Omega))$.

How? Note $\frac{1}{i \partial_{x_j}\varphi} \partial_{x_j} e^{i\varphi} = e^{i\varphi}$; $\frac{1}{i\partial_{\theta_j}\varphi} \partial_{\theta_j} e^{i\varphi} = e^{i\varphi}$

and $d_{(x,\theta)}\varphi \neq 0$.

Consider $\underbrace{\left(\sum \frac{1}{i} \frac{\partial \varphi}{\partial x_j} \partial_{x_j} + |\theta|^2 \sum \frac{1}{i} \frac{\partial \varphi}{\partial \theta_\ell} \partial_{\theta_\ell}.\right)}_{L} \Big/ |\tilde{d}_{x,\theta}\varphi|^2$. $e^{i\varphi} = e^{i\varphi}$

Note. $\varphi, \partial_{x_j}\varphi \in S^1$, $\partial_{\theta_\ell}\varphi \in S^0$ $|\tilde{d}_{(x,\theta)}\varphi|^2 :=$

$$|d_x \varphi|^2 + |d_\theta \varphi|^2 |\theta|^2,$$

$$\rightsquigarrow \quad I_{\varphi,a}(u) = \int e^{i\varphi} ({}^tL)^k (a(x,\theta)u(x)) \, d\theta \, dx.$$

• Take $k$ big and use this (if $a \in S^{-N-1}$ this is fine) as defn of $I_{\varphi,a}(u)$ $(m-k < -N-1)$.