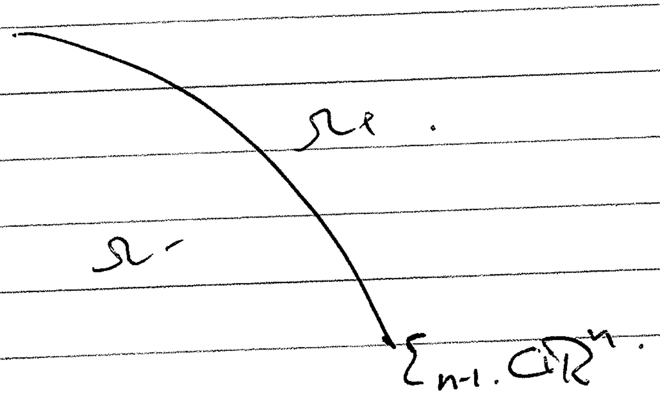
Rafe Lecture 16.

05/11/2012.

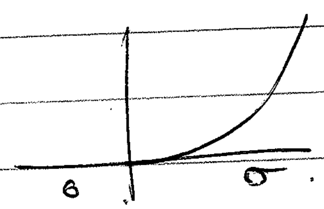
Geometric Optics. Construction.



$$\partial_t^2 u = \Delta u, \quad u\big|_{t=0} = u_0, \quad \partial_t u\big|_{t=0} = u_1.$$

For example: $u_0 = \begin{cases} 1 & \text{on } \Omega_t \\ 0 & \text{on } \Omega \end{cases}$

Guess: $u(t,x) \sim \sum_{j=0}^{\infty} a_j^{\pm}(t,y)\, \lambda_j(S_\pm)$, where $\lambda_j(\sigma) = \frac{\sigma_+^j}{j!}$



$$(\partial_t^2 - \Delta)\big(a_j(t,y)\,\lambda_j(S)\big) = (\partial_t^2 - \Delta)a_j\,\lambda_j(S) + 2\lambda_j'\big(\partial_t S\,\partial_t a_j - \nabla a_j \cdot \nabla S\big) + \lambda_j'\big(\partial_t^2 S - \Delta S\big) + a_j(\lambda_j'')\big(S_t^2 - |\nabla_x S|^2\big)$$

$j = 0$: $\delta''(S(t,x))\underbrace{\big(|\partial_t S|^2 - |\nabla_x S|^2\big)}_{\text{need this to vanish}}$.

Need to choose $S(t,x)$ s.t. $\partial_t S = \pm|\nabla_x S|$ H-J. eq^n.

$S_\pm(t,x) = \pm t + H(x)$ solves H-J. eq^n.

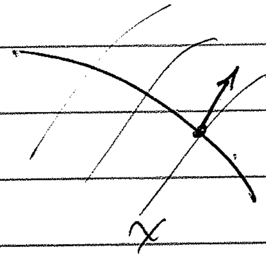
$\iff |\nabla_x H| = 1$.

Note: $\begin{cases} |\partial_t S|^2 - |\nabla_x S|^2 & \text{— Lorentzian.} \\ |\nabla_x H| = 1 & \text{— Riemannian.} \end{cases}$

Observation: $\nabla_x H \cdot \nabla_x H = 1$.

$\Rightarrow$ I) $H^{-1}(c)$ are all regular.

(II). $\partial_s x(s) = \nabla H(x(s))$.



$\Rightarrow$ $\partial_s H(x(s)) \equiv 1$.

$H^{-1}(c_0) = \Sigma$, $\quad H = dist(x, \Sigma)$.

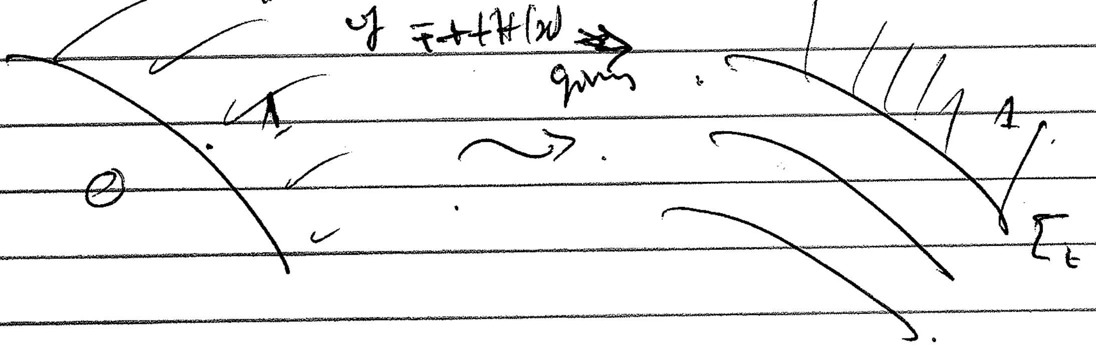
$j=0$, most singular term is

$$2\delta(\Sigma)\left\{\left(-\partial_t a_0 - \nabla_x H \cdot \nabla_x a_0\right) + \frac{1}{2} a_0 (-\Delta H)\right\} = 0.$$

$$\pm \partial_t a_0 + \underbrace{\nabla_x H}_{v(x)} \nabla_x a_0 + \frac{1}{2}(\Delta H) a_0 = 0, \quad a_0|_{t=0} = 1.$$

$\downarrow$

$$\underbrace{\Lambda_0(\mp t + H(x))}_{\text{Heaviside}} \; \underbrace{a_0(t,x)}_{\text{remain smooth in } t}; \quad a_0(0,x) \equiv 1.$$



For higher j:

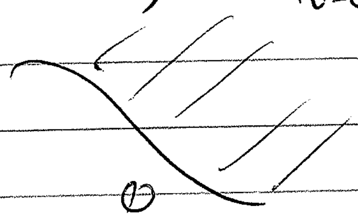
$$2\Lambda_{j-1}(H(x))\left\{\mp\partial_t a_j - \nabla_x H \cdot \nabla_x a_j + \frac{1}{2} a_j \Delta H - (\partial_t^2 - \Delta) a_{j-1}\right\} = 0$$

$$\pm \partial_t a_j + v(x) \nabla_x a_j + \frac{1}{2}(\Delta H) a_j = \Box a_{j-1}.$$

$$a_j|_0 = 0.$$

$$(\partial_t^2 - \Delta) \underbrace{\sum_{j=0}^N a_j^\pm(t,x)\,\lambda_j(\mp t + H(x))}_{u_N} = f_N \in C^N(\mathbb{R}\times\mathbb{R}^{n-1})$$

$\Box u = 0$, $\quad u|_{t=0} = u_0 = A_0\lambda_0(\Omega_+)$, $\quad \partial_t u|_{t=0} = u_1 = A_1\lambda_0(\Omega_+)$.



<u>weather front.</u>

Choose $v \sim \sum a_j^\pm \lambda_j(\mp t + H(x))$ (no diff to choosing a func with given Taylor series).

$\Box v = f \in C^\infty$.

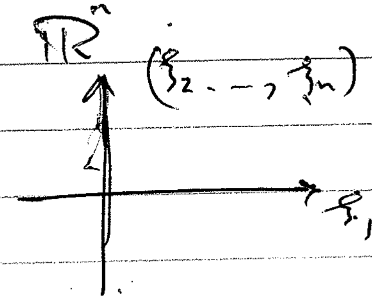
Solve $\Box w = f$, $w|_{t=0} = \partial_t w|_{t=0} = 0$. Then $\underline{u = v - w}$.

$Pu = 0$, (eg. $\Box u = 0$ or $\partial_t u = \pm \Lambda^{1/2} u$).

<u>Q</u>: what can you say about sing spt(u) or $WF(u)$?

<u>Consider.</u> $Pu = \frac{1}{i}\frac{\partial u}{\partial x_1}$ in $\mathbb{R}^n$.

$\sigma_1(P) = \xi_1$



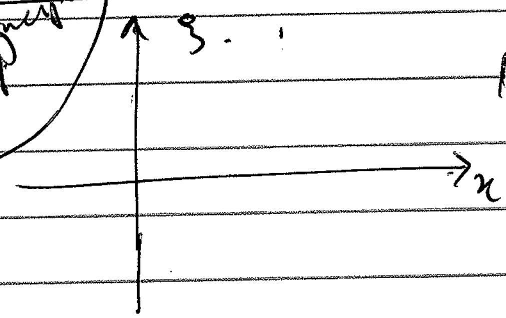
"Real Principal type" — ~~Ran~~ ΨDO order m.
$\sigma_m(P) = p_m$ and $\nabla p_m \neq 0$ when $p_m = 0$.

- For our P_1, we have $\sigma_1(P) = \xi_1 = 0$ for lots of ξ_i, but $\nabla \xi_1 = (1, 0, 0, \dots, 0) \neq 0$.

- $\sigma_2(\square) = \tau^2 + |\xi|^2 = P_2$, $\nabla P_2 = (2\tau, -2\xi)$.

$\boxed{\text{Thm. (Hörmander). If } P_m = 0 \Rightarrow WF(u) \text{ is a union of complete bicharacteristics of } P_m.}$

Real Principal type



$\mathbb{R}^{2n}$. $P_m(x, \xi)$.

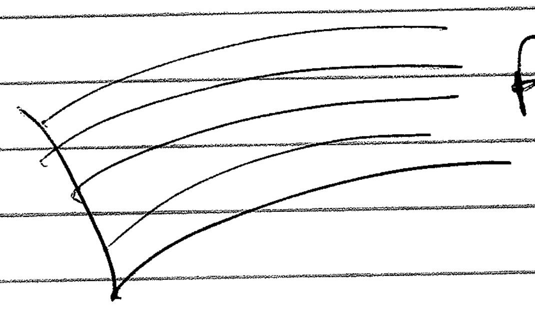
H_{P_m} = Hamiltonian v-field

$$= \frac{\partial P_m}{\partial \xi_j}\frac{\partial}{\partial x_j} - \frac{\partial P_m}{\partial x_j}\frac{\partial}{\partial \xi_j}.$$

$$(x(s), \xi(s)) = \gamma(s).$$

Solve $\gamma'(s) = H_{P_m}(\gamma(s))$.

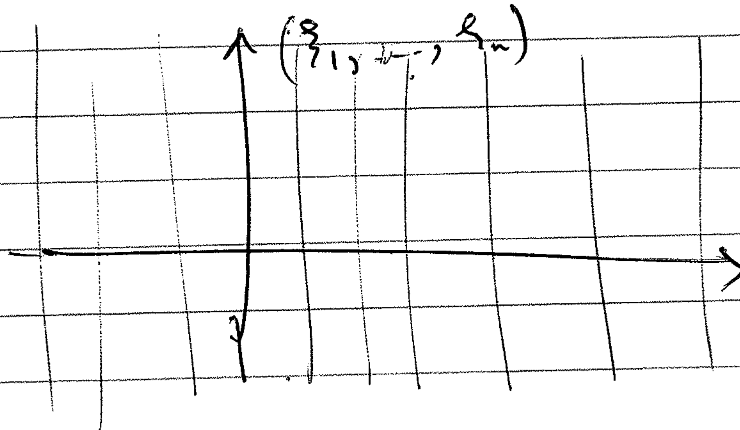
$$\partial_s H(\gamma(s)) = \frac{\partial H}{\partial x_i}\frac{\partial x_i}{\partial s} + \frac{\partial H}{\partial \xi_j}\frac{\partial \xi_j}{\partial s} = 0$$



$P_m = 0$ cuts up space into "null characteristics."

Ie, whenever $(x_0, \xi_0) \in WF$, then its integral curve is in WF.

$Pu = \frac{1}{i}\frac{\partial u}{\partial x_1}$, $p_1 = \xi_1$, $H_{p_1} = \frac{\partial}{\partial x_1}$ ..



$p_1 = 0 = \{(x, 0, \xi_2, \dots, \xi_n)\}$.

$$\frac{\partial u}{\partial x_1} = 0 \Rightarrow u = g(x_2,\dots,x_n).$$

$$WF(u) = \{(x_1, x', \xi_1, \xi') \cdots\}$$
$$= \{(x', \xi') \in WF(g) \quad \xi_1 = 0\}.$$

$\Box u = 0$, $p_2 = 0 \Leftrightarrow \tau = \pm|\xi|$ $\left(p_2 = \frac{1}{2}(\tau^2 - |\xi|^2)\right)$.

$H_{p_2} = \tau\partial_t - \xi_j \partial_{x_j}$

$(t(s), x(s), \tau(s), \xi(s))$. $\dot t = \tau$, $\dot\tau = 0$, $\dot x_j = -\xi_j$, $\dot\xi_j = 0$.

Fix $\tau = \pm|\xi|$. , $\underbrace{t(s) = t_0 + s\tau, \ x(s) = x_0 - s\xi}_{\text{light cone}}$.