Rafe Lec 15. (copied for R Hintz). 02/04/12

$$\begin{cases} Lu = \partial_t u + \sum_{j=1}^n A_j(t,x)\,\partial_{x_j} u + Bu = f & \text{on } \mathbb{T}^n \times \mathbb{R}. \\ u|_0 = g. & \text{(sym. hyp. sys.)} \end{cases}$$

where $A_j^* = A_j$, $u \in L^2(\mathbb{R}\times\mathbb{T}^n; \mathbb{C}^N)$.

Have $L^* = -\partial_t + \sum_{j=1}^n A_j \partial_{x_j} + \tilde{B}$, $\tilde{B} = B + \partial_{x_j} A_j$ i.e.

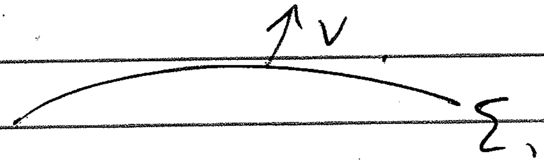
$L^* = -L + C$, $C =$ zeroth order.

Thus, $\langle Lu, u\rangle - \langle u, L^* u\rangle = \frac{1}{i}\int_{\partial O} \langle \sigma_L(t,x,\nu)u, u\rangle$, ~~write~~

write $\lambda = \frac{1}{i}\sigma_L(t,x,\nu)$,

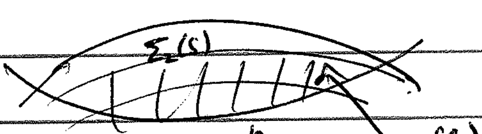
$= 2\,\mathrm{Re}\langle Lu, u\rangle + \langle C^* u, u\rangle$.

Call Σ spacelike w.r.t. L iff



$$\lambda(z,\nu) = \nu_0 I + \sum_{j=1}^n A_j \nu_j \geq 0.$$

Sweeping out a domain.



$$\rightsquigarrow \int_{\Sigma_2(s)} \langle \lambda(z,\nu)u, u\rangle \leq \int_{\Sigma_{1,b}(s)} \langle \lambda(z,\nu)u,u\rangle + C\left(\int_{O(s)} |Lu|^2 + |u|^2\right)$$

with $E(s) = \int_{O(s)} |u|^2$; this gives $E'(s) \leq C_1 E(s) + \underbrace{F(s)}_{\int_{O(s)}|f|^2 + \int_{\Sigma_1}|g|^2}$.

Hence, $\int_{\mathcal{O}(s)} |u|^2 \le C(s-s_0) \int_{\Sigma_1} |g|^2 + C \int_{\mathcal{O}(s)} |f|^2$.

(1) uniqueness of solution.

(2) existence via duality argument with

$$V_T(\mathcal{O}) = \{ v \in C^\infty(\overline{\mathcal{O}}) : v = 0 \text{ on upper bdy} \leadsto v(T,\cdot)=0 \}$$

$0 \le s \le T$, $f \in L^2(\mathcal{O})$, consider

$V_T \ni v \leadsto \langle v, f \rangle$ satisfying

$$|\langle v, f\rangle| \le \|v\|_{L^2} \|f\|_{L^2} \le \|f\|_{L^2} \|L^* v\|_{L^2} \text{; by}$$

Riesz rep, get $u \in L^2$ s.t.

$$\langle v, f\rangle = \langle L^* v, u\rangle \Rightarrow Lu = f \text{ weakly.}$$

Now consider $\begin{cases} \partial_t u = A(t,x,D_x)u + f \\ u|_{t=0} = g \in H^s(\mathbb{R}^n) \end{cases}$ $\quad f \in C^0(\mathbb{R}, H^s(\mathbb{R}^n))$, $A \in C^\infty(\mathbb{R}, \Psi^1(\mathbb{R}^n))$, s.t. $A + A^* \in C^\infty(\mathbb{R}, \Psi^0)$

The "straightforward" approach, treats this as

$\partial_t u = A(t)u + f$, $A(t) \in \mathcal{B}(\mathcal{X})$ fails since $A \in \Psi^1$ (i.e. doesn't map $H^s \to H^s$, say).

- Idea: smooth this out using $J_\varepsilon \in \Psi^{-\infty}(\mathbb{R}^n)$,

$J_\varepsilon = \varphi(\varepsilon D_x)$, $\varphi \in \mathcal{S}(\mathbb{R}^n)$,

$\varphi(0) = 1$, then $J_\varepsilon \to Id$ on Ψ^0. With naive approach, solve

$$\begin{cases} \partial_t u_\varepsilon = J_\varepsilon A J_\varepsilon u_\varepsilon + f \\ u_\varepsilon|_0 = g \end{cases}$$

Want $u_0 = \lim_{\varepsilon \to 0} u_\varepsilon \in C^0(\mathbb{R}, H^s) \cap C^1(\mathbb{R}, H^{s-1})$. with –.

$\Lambda^s = (1+\Delta)^{s/2}$ (in spatial coordinates), we have.
$\|u_\varepsilon(t)\|^2_{H^s} = \|\Lambda^s u_\varepsilon\|^2_{L^2(\mathbb{R}^n)}$.

Compute:

$$\partial_t \|\Lambda^s u_\varepsilon\|_2^2 = 2\operatorname{Re}\langle \Lambda^s J_\varepsilon A J_\varepsilon u_\varepsilon, \Lambda^s u_\varepsilon\rangle + 2\operatorname{Re}\langle \Lambda^s f, \Lambda^s u_\varepsilon\rangle$$

$$= 2\operatorname{Re}\langle A\Lambda^s J_\varepsilon u_\varepsilon, \Lambda^s J_\varepsilon u_\varepsilon\rangle + \underbrace{2\operatorname{Re}\langle [\Lambda^s, A] J_\varepsilon u_\varepsilon, \Lambda^s J_\varepsilon u_\varepsilon\rangle}_{\leq \|u_\varepsilon\|_s^2}$$

$$+ \underbrace{2\operatorname{Re}\langle \Lambda^s f, \Lambda^s u_\varepsilon\rangle}_{\leq \|f\|_s \|u_\varepsilon\|_s}.$$

Have $2\operatorname{Re}\langle A\Lambda^s J_\varepsilon u_\varepsilon, \Lambda^s J_\varepsilon u_\varepsilon\rangle = \langle (A+A^*)\Lambda^s J_\varepsilon u_\varepsilon, \Lambda^s J_\varepsilon u_\varepsilon\rangle$.

$$\Rightarrow \underset{\partial_t \|u_\varepsilon\|_s^2}{\underset{\|}{\partial_t \|\Lambda^s u_\varepsilon\|_2^2}} \leq C\|u_\varepsilon\|_s^2 + C_2 \|f(t,\cdot)\|_s^2.$$

$\Rightarrow \|u_\varepsilon\|^2_{C^0(I,H^s)} \leq C\|f\|^2$., for I cpct interval $y=0$.

Get $u_\varepsilon \to u \in L^\infty(I, H^s) \cap \mathrm{Lip}(I, H^{s-1})$. Take $f_j \in C^\infty(I, H^{s+1})$ with $\|f_j - f\|_{C^0(I,H^s)} \to 0$; get $u_j \in L^\infty(I, H^{s+1}) \cap \mathrm{Lip}(I, H^s)$ and $\|u - u_j\|_{C^0(I,H^s)} \to 0$. Thus, $u \in C^0(I, H^s) \cap C^1(I, H^{s-1})$.

More generally, $\partial_t u = Au + f$, $A \in C^\infty(I, \Psi^1(\mathbb{R}^n))$

is called a symmetrizable hyperbolic system if $\exists$ $S(t,x,D_x) \in C^\infty(\mathbb{R}, \Psi^0)$ and invertible s.t.

$$(SA)^* = -(SA) \mod \Psi^0.$$

Ie, $A^* = -SAS^{*-1}$.

To this, again consider $\begin{cases} \partial_t u_\varepsilon = J_\varepsilon A J_\varepsilon u_\varepsilon + f \\ u_\varepsilon|_0 = g. \end{cases}$

as before $\partial_t \langle \Lambda^s u_\varepsilon ; S \Lambda^s u_\varepsilon \rangle \le \ldots$ (make this work!).

<u>Application</u>

$\partial_t u = Au + f$ s.t. $\sigma_1(A)(t,x,\xi)$ has n distinct purely imaginary eigenvectors $i\lambda_j(t,x,\xi)$.
($\Rightarrow$ smoothly diagonalisable for $\xi \neq 0$).

The eigenspaces of $\sigma_1(A)(t,x,\xi)$ depend smoothly on (t,x,ξ). (prove this!). Namely, $(\sigma_1(A)(t,x,\xi) - \lambda)^{-1}$ has simple poles at $\lambda = i\lambda_j$, and
$\frac{1}{2\pi i}\oint (\sigma_1(A)(t,x,\xi) - \lambda)^{-1} d\lambda = P_j$ projects onto j^{th}

eigenspace off all the k^{th} eigenspaces $k \neq j$.

Take $S = \sum P_j^* P_j$ (i.e., principal symbols agree), and noting that $A = \sum i\lambda_k P_k$, we have
$P_j A = i\lambda_j P_j$, $P_j^* P_j A = i\lambda_j P_j^* P_j$ and
$A^* P_j^* P_j = -i\lambda_j P_j^* P_j$ and thus $\sum P_j^*(P_j A) + \sum (A^* P_j^*) P_j = 0$ mod Ψ'.

<u>Explicitly</u>, & as an example: $(\partial_t^2 - \Delta)u = f$.

Rewrite this as $\partial_t u = v$, $\partial_t v = \Delta u + f$, ie
$$\partial_t \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \Delta & 1 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} 0 \\ f \end{pmatrix}.$$

Since $\begin{pmatrix} 0 & 1 \\ \Delta & 0 \end{pmatrix}$ is not 1st order, rewrite.

$$\partial_t u = \sqrt{-\Delta}\, v, \quad \partial_t v = -\sqrt{-\Delta}\, u + \underbrace{(-\Delta)^{\frac{1}{2}}}_{} f.$$

here Δ has problems on $\mathbb{T}$ (const function).

Then $\partial_t^2 u = \sqrt{-\Delta}$, $\partial_t v = \Delta u + f$. I.e., get

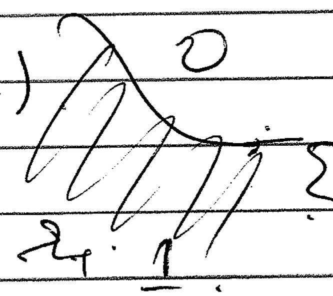
$$\partial_t \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 & \sqrt{-\Delta} \\ -\sqrt{-\Delta} & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} 0 \\ (-\Delta)^{\frac{1}{2}} f \end{pmatrix}.$$

$$\sigma_1(\ldots) = \begin{pmatrix} 0 & |\xi| \\ -|\xi| & 0 \end{pmatrix}, \text{ eig. } \pm i|\xi|, \text{ hence}$$

have symm. hyp. system.

<u>Propagation of singularities, geometric optics.</u>

$\Box = \partial_t^2 - \Delta$, consider $\begin{cases} \Box u = 0 \\ u|_{t=0} = H_{\Omega_+} \text{ (heaviside)} \\ \partial_t|_{t=0} = 0. \end{cases}$



<u>Idea</u> (geometric optics).

$$u \sim \sum a_j(t,x) \left(\frac{\tau_+^j}{j!} \right) s(t,x)$$

where $\tau_+^j = \begin{cases} \tau^j, & \tau > 0 \\ 0, & \tau \leq 0 \end{cases}$, $j > -1$.

Have $\partial_\tau \tau_+^j = j\,\tau_+^{j-1}$ ⇝ get $\tau_+^\alpha / \Gamma(1+\alpha)$,

(make sense with some. $\partial_\tau(u^{(k)}) = u^{(k-1)}$.
use this to define $u^{(\alpha-1)}$: Eg $u^{(-1)} = \delta$)

hol family of distributions on $\mathbb{R}$, in $\alpha \in \mathbb{C}$.

Plug this into eq^n and hope for the best.

Compute
$$(\partial_t^2 - \Delta)\,(a\,\chi(s)) = (\partial_t^2 - \Delta)a \cdot \chi(s) + 2\left[\partial_t a \cdot \partial_t(\chi(s)) - \nabla_x a \cdot \nabla_x \chi(s)\right] + a\left(\chi'(s)\,\partial_t^2 s + \chi''(s)\,(\partial_t s)^2 - \chi'(s)\,\Delta_x s - \chi''(s)\,|\nabla_x s|^2\right).$$

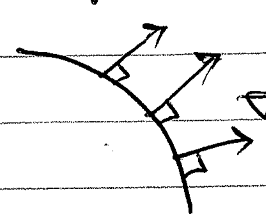
with $\chi = \frac{\tau_+^j}{j!}$, $\chi^{(\ell)} = \frac{\tau_+^{j-\ell}}{(j-\ell)!}$. Expand this out.

Only look at most singularity here, for $j=0$.

$$\delta'(s)\,\underbrace{\left[|\partial_t s|^2 - |\nabla_x s|^2\right]}_{\text{Eikonal equation}} = 0 \quad \text{(nothing can cancel it!)}.$$

Study Eikonal eq^n as $|\partial_t s|^2 = |\nabla_x s|^2$, ie $\partial_t s = \pm |\nabla_x s|$.
(→ Hamilton-Jacobi theory).

Ansatz: $s_\pm(t,x) = \pm t + H(x)$, thus $s_- = H(x) - t$, E eqn means $1 = |\nabla_x H|$. Look at level sets of such H.



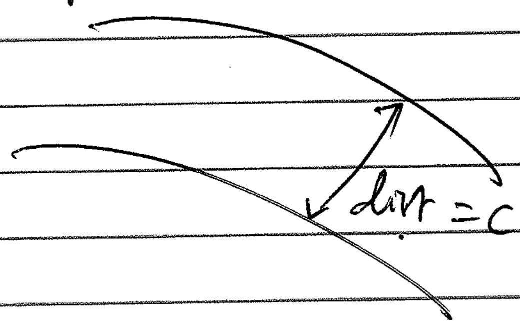
∇H. Take flow lines $x(s)$, $\dot{x}(s) = \nabla H(x(s))$.

$$\Rightarrow \frac{d}{ds} H(x(s)) = \nabla H(x(s)) \cdot \dot{x}(s) = |\nabla H|^2 = 1.$$

$\rightsquigarrow$ flow lines move with speed 1.

Thus $\{H^{-1}(C)\}$ are equidistant surfaces.

Hence for $s = t - H(x)$,



$\Sigma_{t=c} = \{s = 0 \text{ at } t = c\}$.

$\Sigma_{t=0} = \{s = 0 \text{ at } t = 0\}$