Rafe. Lecture 13. 26/10/2012.

$\Box$ d'Alembertian. $= \{\Delta$ for Lorentzian metric$\}$.

$\Box = D_t^2 - \Delta$ on $\mathbb{R} \times M_g$, $h = -dt^2 + g$.

More generally, $\Box_h$ for $h = h_{ij}\, dx^i dx^j$ sig. $\begin{pmatrix} -1 & & 0 \\ & +1 & \\ 0 & & \ddots\ +1 \end{pmatrix}$.

1) Conservation of energy

$$E(t) = \frac{1}{2} \int_{t=\text{const}} \left(u_t^2 + |\nabla_x u|^2 \right) dx \quad \text{const. provided}$$

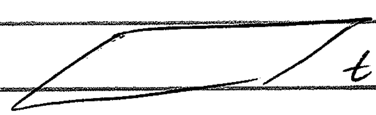
$u(t,\cdot)$ decays sufficiently.

2). Finite Propagation Speed.

(N^{n+1}, h) Lorentzian.

S hypersurface is called spacelike if $h(\nu,\nu) < 0$ for ν unit normal to S.

Lor. $h = -dt^2 + g$ situation, $S = \{t = f(x)\}$.



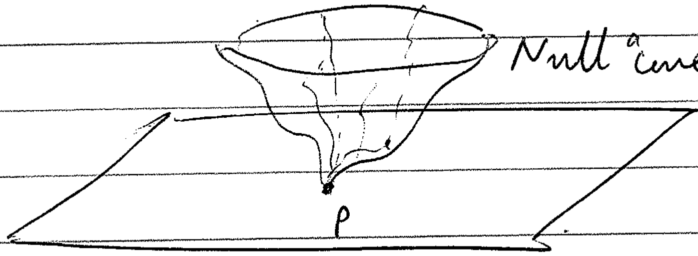
$t = 0$. $\nu = (-1, -\nabla_x f)$.

$|\nabla_x f| < 1$.

Null geodesics $\gamma(s)$ $h(\gamma'(s), \gamma'(s)) \equiv 0$.

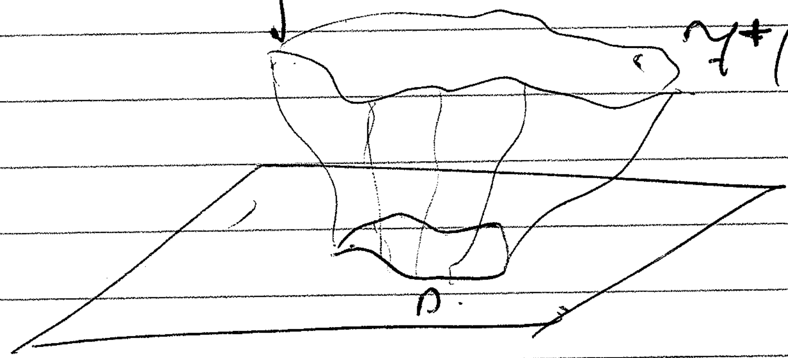
Eg. $\gamma(s) = (s, \alpha(s))$, $\alpha(s)$ geodesic for g.

$\gamma' = (1, \alpha'(s))$, $h(\gamma', \gamma') = -1 + |\alpha'(s)|^2 = 0$.



$$\left\{\begin{array}{l} \Box u = f \\ u|_S = g_0, \\ \partial_\nu u|_S = g_1. \end{array}\right\}$$ typical problem we solve.

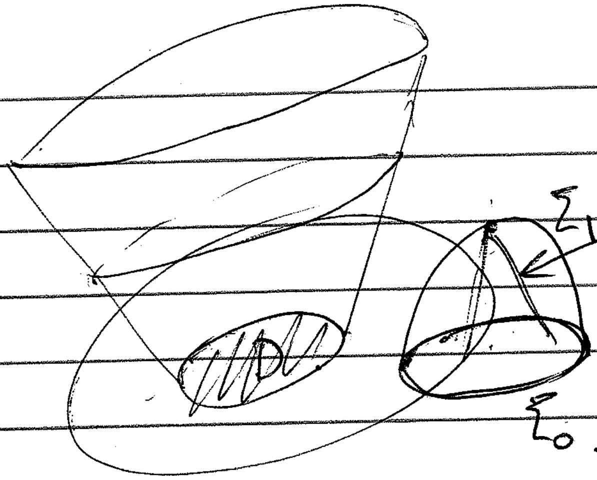
DCS.



Prop. Suppose $(N, -dt^2 + g)$ is the background and $\Box u = 0$, $\mathrm{spt}(g_0, g_1) \subset D \subset S'$, spacelike. $S = \{t = 0\}$, then

$$\mathrm{spt}\, u(t, \cdot) \subset D_t = \{ x \in S \times \{t\}; d(x, D) \leq t \}.$$

Ie, support grows at speed 1.

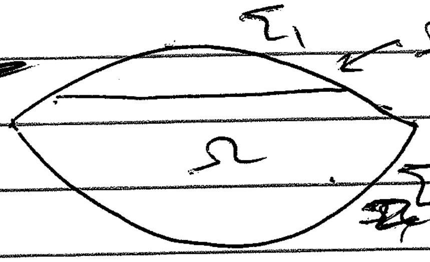


$\Omega = \bigcup \Sigma_t \qquad 0 \le t \le 1.$

Σ_t spacelike.

If $\Sigma_0 \cap D = \emptyset$, then foliation by spacelike Σ_t to ~~obt~~ obtain Ω leaves it disjoint from $J^+(D)$

Pf



$$0 = \int_\Omega u_t (u_{tt} - \Delta u)$$

$$= \int_\Omega \partial_t \left(\tfrac{1}{2} u_t^2\right) + \partial_x u_t \cdot \partial_x u.$$

$$- \int_\Omega \mathrm{div}_x (u_t \nabla_x u).$$

$$= \int_\Omega \left\{ \partial_t \left(\tfrac{1}{2} u_t^2 + |\nabla_x u|^2\right) - \mathrm{div}_x (u_t \nabla_x u) \right\}$$

$$= \frac{1}{2} \int_{\partial\Omega} \left(u_t^2 + |\nabla_x u|^2\right) \nu_t \, dS - \frac{1}{2} \int_{\partial\Omega} 2 u_t \, \partial_{\nu_x} u \, |\nu_x| \, dS$$

$$= \frac{1}{2} \int_{\partial\Omega} \left\{ \left(u_t^2 + |\nabla_x u|^2\right) \nu_t - 2 u_t \, \partial_{\nu_x} u \, |\nu_x| \right\} dS.$$

Σ_i spacelike $\Rightarrow \quad -\nu_t^2 + |\nu_x|^2 < 0 \iff |\nu_t| > |\nu_x|$.

$$\int_{\Sigma_1} \left(u_t^2 + |\nabla_x u|^2\right) \nu_t - 2 u_t \, \partial_{\nu_x} u \, |\nu_x| = \int_{\Sigma_0} \left(u_t^2 + |\nabla_x u|^2\right) \nu_t - 2 u_t \, \partial_{\nu_x} u \, |\nu_x|$$

C-S.

$$\Rightarrow \quad c_1 \int_{\Sigma_1} (u_t^2 + |\nabla_x u|^2) \le c_0 \int_{\Sigma_0} (u_t^2 + |\nabla_x u|^2).$$

Hence if $u_t = \nabla_x u = 0$ on $\Sigma_0 \Rightarrow u_t = \nabla_x u = 0$.

Now suppose $\Omega = \bigcup \Sigma_t$, all spacelike curves. Then for any such foliations, we can apply this and to $u = 0$ on $\Sigma_0 \Rightarrow u \equiv 0$ on Ω.

If $\Box u = 0$, Define: $T = du \otimes du - \frac{1}{2}|du|^2 h$

stress-energy tensor.

Fact: $T = (T^{jk})$, is divergence-free.
Exercise (uses only $\nabla h = 0$ and $\Box u = 0$).

Z = killing Field. $\Leftrightarrow$ ∇Z skew ($\Leftrightarrow$ $\mathcal{L}_Z h = 0$).

Claim ~~$\mathcal{L}_Z h = 0$~~ $v \mapsto T(Z, v)$ is divergence free.

$$(T^{jk} Z_k)_{;j} = T^{jk}_{;j} Z_k + T^{jk} Z_{k;j} = 0.$$

$$X = Z \llcorner T \quad (X = v \mapsto T(Z, v)).$$

Short pf of before:

$$0 = \int_\Omega \operatorname{div}(X) = \int_{\partial\Omega} X \cdot \nu = \int_{\Sigma_1} T(Z, \nu) - \int_{\Sigma_0} T(Z, \nu)$$

$T = du \otimes du - \frac{1}{2}|du|^2 h.$

"Energy" $\to$ $T(Z,\nu) = \Sigma_{Z,\nu}(du) = (Zu)(\nu u) - \frac{1}{2}\underbrace{\left(-u_t^2 + |\nabla_x u|^2\right)}_{(-dt^2+g)(Z,\nu)}$

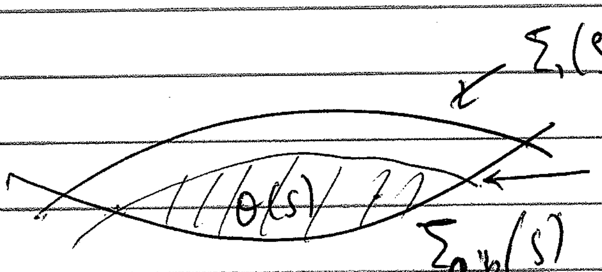
C.S. we want is $\frac{1}{2} h(Z,\nu)\langle du, du\rangle_h \le (Zu)(\nu u)$.

$Lu = \Box u + X\cdot u = f.$

~~Before~~

My note: For Δ, we have $C^\alpha \to H^0$ estimates. Here, we lose a derivative $H^1 \to H^0$ estimates. This is because singularities are not smoothed & but rather propagated. But ~~this~~ a "creation" of singularities is very limited $\to$ constrained to light cone bdry.

(s variable} in t)



$X = T(Z,\cdot)$, as before, but no longer div-free.

$$\operatorname{div} X = T^{jk}{}_{;k} Z_j + T^{jk} Z_{j;k}$$

$$= \langle \operatorname{div} T, Z\rangle + \langle T, \nabla Z\rangle.$$

bound of killing, for Z time like.

$$|\langle T, \nabla Z\rangle| \leq C\, E_{Z,Z}(du).$$

$$\operatorname{div} T = \nabla u \cdot \Box u = \nabla u\,(f - Xu).$$

$$\Rightarrow |\langle \operatorname{div} T, Z\rangle| \leq C' E_{Z,Z}(du) + k_1 |u|^2 + k_2 |f|^2.$$

$$\int_{\Sigma_1(s)} E_{Z,\nu_2}(du) \leq \int_{\Sigma_{0,b}(s)} E_{Z,\nu_1}(du) + C\left(\int_{O(s)} E_{Z,Z}(du) + |u|^2 + |f|^2\right)$$

$$\boxed{\Box u = f;\ u|_{\Sigma_0} = g,\ du|_{\Sigma_0} = w}$$

$$\int_{O(s)} |u|^2 \leq C \int_{\Sigma_{0,b}(s)} |g|^2 + C \int_{O(s)} E_{Z,Z}(du).$$

$$E(s) = \int_{O(s)} E_{Z,Z}(du).$$

$$E'(s) \leq \int_{\Sigma_2(s)} E_{Z,Z}(du) \leq C\, E(s) + F(s).$$

F defined by g, w, initial data.

$$\Rightarrow \left(e^{-Cs} E(s)\right)' \leq e^{-Cs} F(s) \Rightarrow e^{-Cs} E(s) \leq \int_{s_0}^{s} e^{-C\sigma} F(\sigma)\, d\sigma.$$

Philosophy: Consider such evolution equations as ODE in a space.

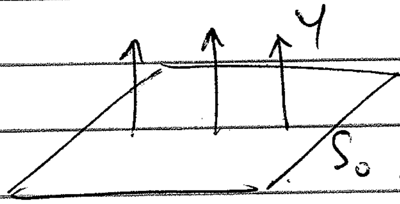
$\rightsquigarrow$ Not exactly, because A is unbounded.

$$\Rightarrow \int_{O(s)} E_{2,t}(du) \le C(s-s_0)\int_{\Sigma_1} |g|^2+|w|^2 + e\int_{O(s)} |f|^2.$$

Cor:

Existence? $Lu = f \quad (L = \square + X).$

$u|_{S_0} = g_0, \quad Yu|_{S_0} = g_1$



We've proved:

$$\|u\|^2_{H^1(O)} \le C\left(\|Lu\|^2_{L^2(O)} + \|g_0\|^2_{H^1(S_0)} + \|g_1\|^2_{L^2(S_0)}\right)$$

$$V_T(O) = \{w \in C^\infty(\bar{O}) : w = dw = 0 \text{ on } S_T\}.$$

Apply for $L^* = \square + X^*$ backwards on V_T.

$$\Rightarrow \|v\|^2_{H^1(O)} \le C\|L^* v\|^2_{L^2(O)}.$$

Solve $Lu = f$, $u|_{S_0} = Y|_{S_0} = 0$.

$$\langle u, L^* v\rangle = \langle f, v\rangle, \quad \forall v \in V_T(O).$$

$v \longmapsto \langle f, v\rangle$. bdd. $|\langle f,v\rangle| \le C\|v\|^2_{H^1} \le C'\|L^* v\|^2$

Riesz Rep: $\rightsquigarrow$ $\langle f, v\rangle =. \langle u, L^* v\rangle$ $u \in L^2$.

Apriori $u \in L^2 \Rightarrow Lu \in H^{-2}$.