Rafe Lec. 12. 24/10/2012.

$L_V = -\Delta + V$, $\Omega \subset \mathbb{R}^n$ (cannot be mfld).

N_V: Dirichlet-to-Neuman op.

$N_V(\varphi) = \psi$ if $L_V u = 0$, $u|_{\partial\Omega} = \varphi$, $\partial_\nu u|_{\partial\Omega} = \psi$.

Prbm: Is $V \to N_V$ injective?

Thm. (Sylvester-Uhlmann '86). Yes, if $n \geq 3$.

$(N_{V_1} = N_{V_2} = N \in \Psi^1(\partial\Omega)$, elliptic, self-adj$)$.

$0 = \int u(L_V v) - (L_V u)v = \int u_0 N_V(v_0) - N_V(u_0)v_0$.

$\sigma_1(N) = |\eta|$.

$L_1 u_1 = 0$, $L_2 u_2 = 0 \Rightarrow 0 \int_\Omega (L_1 u_1)u_2 - u_1(L_2 u_2)$
$= \int (V_1 - V_2) u_1 u_2$.

Goal. Show that $\{u_1 u_2 : L_j u_j = 0 \text{ in } \Omega\}$ is ~~dense~~ dense. $\Rightarrow V_1 - V_2 = 0$.

$(-\Delta + V)(e^{x\cdot\zeta}(1+w)) = 0$, $\zeta \in \mathbb{C}^n$, $\zeta\cdot\zeta = 0$.

$-2\nabla(e^{x\cdot\zeta}) \cdot \nabla(1+w) - e^{x\cdot\zeta}\Delta(1+w) + Ve^{x\cdot\zeta}(1+w) = 0$

$\Delta w + 2\zeta\nabla w + Vw = -V$.

① $(\Delta + \zeta\cdot\nabla)u = f$.

$$\|u\|_{L^2_\delta} \leq \frac{C}{|\zeta|}\|f\|_{L^2_{1+\delta}} \qquad -1 < \delta < 0.$$

$$\int |u|^2 (1+|x|^2)^\delta \, dx.$$

② Perturb to solve $(\Delta + 2\zeta\cdot\nabla + V)w = f$.

$$\|w\|_{L^2_\delta} \leq \frac{C}{|\zeta|}\|f\|_{L^2_{\delta+1}}.$$

③ Since ① is a const. coeff. eq^n. Can take as many derivatives as we like. So, we get

$$\|u\|_{H^m_\delta} \leq \frac{C}{|\zeta|}\|f\|_{H^m_{\delta+1}}.$$

$$e^{x\cdot\zeta}\left(1 + O\left(\frac{1}{|\zeta|}\right)\right).$$

Pick $\xi \in \mathbb{R}^n$, $\xi \neq 0$, Choose $\eta, r \perp \xi$. $\eta \perp r$.

$|\xi \pm \eta| = |r|$.

$$\zeta_1 = r + i(\xi + \eta), \quad \zeta_2 = -r + i(\xi - \eta).$$

$$u_1 = e^{x\cdot\zeta_1}(1+w_1); \quad u_2 = e^{x\cdot\zeta_2}(1+w_2).$$

$$u_1 u_2 = e^{2i\xi}(1+w_1)(1+w_2)$$

Philosophy: $e^{ix\cdot\xi}$ form dense set, ~~need to show~~ is to show we can approximate $e^{ix\cdot\xi}$ by solutions

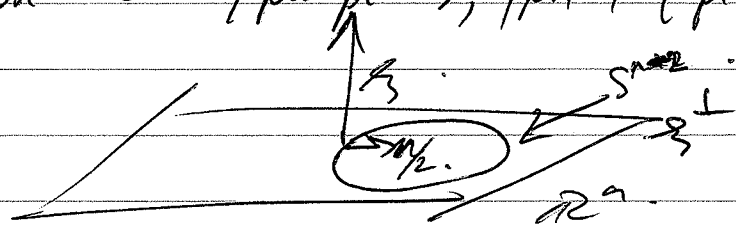
F.T. variable μ, $\zeta = \xi + i\eta$.

Then $(\Delta + \zeta\cdot\nabla) v = f$ FT is

$$(-|\mu|^2 + i(\xi + i\eta)\cdot\mu)\hat{u} = \hat{f}.$$

$$\ell(\mu) = -|\mu|^2 - \eta\cdot\mu + i(\xi\cdot\mu).$$

$\ell(\mu) = 0$ on $S^{n-2} = \{\mu : \mu \perp \xi,\ |\mu|^2 + \eta\cdot\mu + \frac{|\eta|^2}{4} = \frac{|\eta|^2}{4}\}$.



Note: $\ell(\mu)$ vanishes simply on S^{n-2}; i.e. $\nabla\ell \neq 0$ there.

$$f \to \mathcal{F}^{-1}\left(\frac{\hat{f}}{\ell(\mu)}\right).$$ $\underbrace{\text{"same as"}}_{\text{through power of}}$ microlocal anal.

$$f \longmapsto \mathcal{F}^{-1}\left(\frac{\hat{f}}{\mu_1 + i\mu_2}\right)$$; inverse C-R op.

$\Delta : L^2_\delta \longrightarrow L^2_{\delta+2}$. Fredholm. $-1 < \delta < 0$.

$\partial_{\bar z} : L^2_\delta \to L^2_{\delta+1}$. I.e., $\underbrace{\|u\|_{L^2_\delta} \le C\|f\|_{L^2_{\delta+1}}}$

fundamental op $\partial_{\bar z} = f$. $-1 < \delta < 0$.

Lemma: $\varphi : \mathcal{U} \to \mathcal{V}$ bdd open sets in $\mathbb{R}^n$.
$|D\varphi| \le C$, $|D\varphi^{-1}| \le C$.

~~Suppose~~ If $\hat{f}$ sptd in $\mathcal{V}$. $\Phi f = \mathcal{F}^{-1}(\hat{f}\circ\varphi)$.
$\hat{g}$ sptd in $\mathcal{U}$ $\Phi^{-1} g = \mathcal{F}^{-1}(g\circ\varphi^{-1})$ satisfy
$\|\Phi f\|_{L^2_\delta} \le C \|f\|_{L^2_\delta}$.

- $\delta = 0$, $\|\Phi f\|_{L^2} = \|\widehat{\Phi f}\|_{L^2}$
$= \|\hat{f}\circ\varphi\|_{L^2}$
$= \int |\hat{f}\circ\varphi|^2 = \int |\hat{f}|^2 \cdot |D\varphi|$
$\le C \int |\hat{f}|^2 = C \int |f|^2$.

- $\delta = 1$, $\|\Phi f\|^2_{L^2_1} = \int |f(x)|^2 (1+|x|^2) = \sum_{|\alpha|\le 1} |D^\alpha \hat{f}|^2$.

$\|\Phi f\|^2_{L^2_1} = \int |\hat{f}\circ\varphi|^2 + \sum_j |\partial_{x_j}(\hat{f}\circ\varphi)|^2 \le C \|f\|^2_{L^2_1}$.

- $\delta = -1$, $g \in L^2_1$, $\operatorname{spt} g \in \mathcal{U} = \operatorname{spt}(\widehat{\Phi f})$.

$\langle \Phi f, g\rangle = \int (\hat{f}\circ\varphi)\,\hat{g} = \int \hat{f}\,(\hat{g}\circ\varphi^{-1})\,|D\varphi|$.
$\le \|f\|_{L^2_{-1}} \|\Phi^{-1} g\|_{L^2_1}$.

$\le C \|f\|_{L^2_{-1}} \cdot \|g\|_{L^2_{+1}}$.

Then interpolation theory, we get this for $-1 \le \delta \le 1$.

Lemma If $u \in \mathcal{S}' \cap L^2_{loc}$ and

$$\limsup_{R\to\infty} \frac{1}{R^k} \int_{|x|\le R} |u|^2 < \infty \quad \text{in } \mathbb{R}^n.$$

$$\hat{u} = \hat{u}_0 \, \delta_{\mathbb{R}^{n-k}} (\xi'')$$

$$u(x', x'') = u_0(x') \cancel{1_{x''}} \qquad u_0 \in C(\mathbb{R}^{n-k})$$

motivation: Suppose $\operatorname{spt} \hat{u} \subset Y^{n-k}$, C^1 submanifold.

Then, $\hat{u} = \hat{u}_0 \, \delta_Y$, $\hat{u}_0 \in C(Y)$.

$$\int |\hat{u}_0| \, dS_Y \lesssim \limsup_{R\to\infty} \frac{1}{R^k} \int_{|x|\le R} |u|^2.$$

Cor. $u \in L^2_\delta$, $-1 < \delta < 0$, $\hat{u}$ spted on Y^{n-2}.
$\Rightarrow u \equiv 0$.

Pf. $$\|u\|^2_{L^2_\delta} \ge \int_{|x|\le R} (1+|x|^2)^\delta |u(x)|^2 \, dx$$

$$\ge R^{2\delta} \int_{|x|\ge R} |u(x)|^2$$

precisely why $\delta < 0$

we need so far exp is -ve. $\rightarrow$ $$R^{-2-2\delta} \|u\|^2_{L^2_\delta} \ge R^{-2} \int_{|x|\le R} |u|^2.$$

growth -1 0 decay δ axis

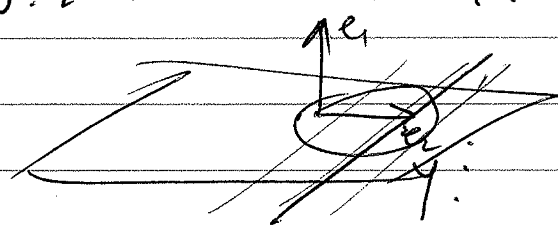
$$\int |u|^2 (1+|x|^2)^{\delta} < \infty$$

By moving δ positive, makes function space smaller and smaller. This here says that as long as $\delta > 1$, still small space.

Dually, we have surjection for larger δ. In fact $-1 < \delta < 0$ is exactly where injective and surjective holds.

Assume $\xi = s(e_1 + i e_2)$, $Y = S^{n-2}$.

$N_\varepsilon(Y) = \varepsilon$-nbhd of Y. Cover Y by $\{U_j\}_{j=2}^{n}$, $U_2 = Y \cap \{(x_2 - s/2)^2 \geq \frac{s^2}{5n}\}$.



$U_j = \{ x_j^2 \geq \frac{s^2}{5n} \}$, $j \geq 3$.

$V_1 = \mathbb{R}^n \setminus N_{\varepsilon/2}(Y)$, $V_j = N_\varepsilon(U_j)$.

$\{B_j\}$ p.o.u. subordinate to $\{V_j\}$.

$$\hat{f} = \sum \rho_j \hat{f} = \sum \hat{f}_j .$$

$j \geq 2$, $\varphi_j : \mathcal{U} \cap \{\text{spt } \rho_j\} \longrightarrow \mathbb{R}^{n-2}$.

<u>Need</u> φ_j to have held gradient.

$(G_j(\mu)\cdot)_i$, $i = 1, \ldots, n$.

$i \neq 2$, or j.

$$G_{jj}(\mu) = \frac{\mu_1^2 + (\mu_2 - s/2)^2 + \cdots + \mu_n^2 - s^2/4}{2s}.$$

$$G_{j2}(\mu) = \mu_2 - s/2.$$
$$G_{ji}(\mu) = \mu_i.$$
$$\hat{w}_j = \frac{1}{s}\left(\frac{f_j \circ G_j}{\mu_j + i\mu_1}\right). \quad j \neq 1.$$

(Apply here (1) and (2)).

$j = 1$.

$$\|u_1\|_{L^2_s}^2 \leq \|u_1\|_{L^2}^2 = \int \frac{|\hat{f}|^2 \cdot \rho_1^2}{\ell(\mu)} e^{i\mu x} d\mu.$$

$$\leq \frac{1}{s} \cdot \|f_1\|_{L^2}^2.$$

$$\leq \frac{1}{s^2} C \|f_1\|_{L^2_{s+1}}^2.$$

$\boxed{\text{Solves unperturbed}}$

Next: Real problem $(\Delta + 2\varsigma\cdot\nabla)w + Vw = -V$

S = soln operator for $\Delta + 2\varsigma\cdot\nabla$.

Then $(I + SV)w = S(-V)$

$$\|S\|_{L^2_{\delta+1}\to L^2_\delta} \le \frac{C}{|\varsigma|}.$$

$$\|SV\|_{L^2_\delta\to L^2_\delta} \le \frac{C}{|\varsigma|} \Rightarrow (I+SV) \text{ invertible if chose } |\varsigma| \text{ large!}$$

$$w = (1+SV)^{-1}(S(-V)).$$

(*). Can't do this for even submanifolds, because we don't have the luxury of these dense exp. functions.

Next time: Hyperbolic.

$$D_t^2 - \sum_{j=1}^n D_{x_j}^2 \text{ in } \mathbb{R}^{n+1}.$$

$\mathbb{R}\times M$, $h = -dt^2 + g$, (M,g) Reimann.

(1) Conservation of energy for $\Box_g = -\partial_t^2 + \Delta_g$.

(Note: $\sum h^{ij}\partial^2_{w_i w_j}$ $(w_0,\dots,w_n)$. h_{ij} signature $(1,n)$).

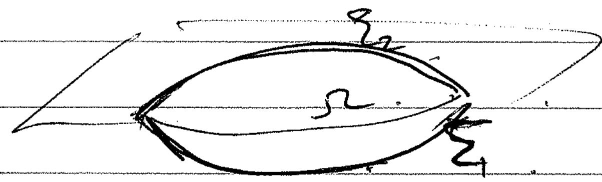
$\sim \sum_{i=1}^n \partial^2_{w_i} - \partial^2_{w_0}$ (diagonalize).

$$E(u(t)) = \frac{1}{2}\int_M |\partial_t u(t,\cdot)|^2 + \frac{1}{2}|\nabla_x u(t,\cdot)|^2.$$

Th^{m}. If $\Box u = 0 \Rightarrow E' = 0$.

Pf.
$$E' = \int u_t u_{tt} + \nabla_x u \cdot \nabla_x u_t$$
$$= \int u_t \underbrace{(u_{tt} - \Delta_g u)}_{\Box u}.$$

② Localisation ⇝ Finite prop. speed.



Lens shaped domain.

Find $c_1 \int_{\Sigma_1} |u_t|^2 + |\nabla_x u|^2 \simeq c_2 \int_{\Sigma_2} |u_t|^2 + |\nabla_x u|^2$.