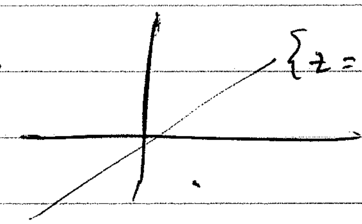
Rafe Lecture 11. 22/10/2012.

~~This is precisely.~~

$$\begin{cases} Pu = f \\ B(\gamma u) = h \end{cases}$$

E.g. for $P = \Delta \rightsquigarrow C_P = \begin{pmatrix} \frac{1}{2} & \frac{1}{2i|\mu|} \\ \frac{i}{2}|\mu| & \frac{1}{2} \end{pmatrix}$.

w/ range



$\{z = i|\mu| t z_0\}$.

Need hypotheses. B when restricted to C_P;

Dirichlet. $B_D \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} = u_0$,

Neumann $B_N \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} = u_1$.

When is $B(\gamma u) = A_0 u_0$ ($A_0 \in \Psi^k(\partial\Omega)$ good?).

If $\sigma(A_0) \neq 0$, i.e. A_0 elliptic.

$B(\gamma u) = A_0 u_0 + A_1 u_1 \rightsquigarrow \sigma(A_0) + i|\mu|\sigma(A_1) \neq 0$.

Bad e.g. $\alpha_1 + i|\mu|\alpha_0 \neq 0$, e.g. $\alpha_0 = 1$, $\alpha_1 = -i|\mu|$.

corresponds to $-i\sqrt{\Delta_0}\, u_0 + u_1 = h$.

(Reminder $u_1 = \frac{1}{i}\partial_\nu u|_{\partial\Omega}$).

Or $A_0 = i\mu_j$, $A_1 = 1$. $i\mu_1 + i|\mu|$ is <u>not elliptic</u> $\rightsquigarrow \partial_{y_1} u_0 + u_1 = 0$, i.e.

$\frac{\partial u}{\partial \nu} + i v_0 u = h$.

{ Going back to C_P, $P = \Delta$, recall: If $\Delta u = 0$ in Ω, then (u_0, u_1) are related: $\hat{u}_1 = i|\mu|\hat{u}_0$.

$$\frac{\partial \hat{u}}{\partial \nu} = -|\eta|\,\hat{u}\big|_{\partial\Omega}.$$

I.e. given $\varphi \in H^s(\partial\Omega)$, solve $\begin{cases} \Delta u = 0 \\ u|_{\partial\Omega} = \varphi \end{cases}$

Define $N(\varphi) = \frac{\partial u}{\partial \nu}\Big|_{\partial\Omega}$, Dirichlet-to-Neumann operator.

Got: $N \in \Psi^1(\partial\Omega)$, $\sigma_1(N) = -|\eta| \Rightarrow N$ elliptic.

Thm: $(\Omega, g) \rightsquigarrow N_g \rightsquigarrow \sigma(N_g)$ = full symbol determines entire series of g at $\partial\Omega$.

Calderon Inverse Problem: Does N determine Ω or g or Δ? Want to study: is $g \mapsto N_g$ injective. I.e. is g determined by N_g. (No word about practical issues of actually recovering g...).

Fix $\Omega \subset \mathbb{R}^n$, smooth, bdd, conn'ct. For $V \in C^\infty(\bar{\Omega})$, put $L_V = -\Delta + V$. Define N_V. Q. does N_V determine V? I.e. Is $V \mapsto N_V$ injective?

Thm (Sylvester-Uhlmann) If $n \geq 3$, this is true!

Want to prove this: $V_1, V_2 \rightsquigarrow N_1, N_2$. Q. $N_1 = N_2 \Rightarrow V_1 = V_2$.

Delux. www.knitkits.com.

Suppose $L_j u_j = 0$, $j = 1, 2$, $u_j|_{\partial\Omega} = \varphi_j$.

$$\Rightarrow 0 = \int_\Omega (-\Delta + V_1)u_1 u_2 - u_1(-\Delta + V_2)u_2 .$$

$$= \int_\Omega (V_1 - V_2)u_1 u_2 + \int_{\partial\Omega} \left(u_1 \frac{\partial u_2}{\partial \nu} - \frac{\partial u_1}{\partial \nu} u_2\right)$$

$$\parallel$$

$$\int_{\partial\Omega} \varphi_1 N(\varphi_2) - N(\varphi_1)\varphi_2 .$$ using $N = N_1 = N_2$

N is self-adj (follows from this argument with $V_2 = V_1$), hence get $\int_\Omega (V_1 - V_2)u_1 u_2 = 0$.
If we knew that $\{u_1 u_2 : L_1 u_1 = 0 = L_2 u_2\}$ is dense in $C^0(\Omega)$, we can conclude that $V_1 = V_2$.

Idea: For $V = 0$, $e^{x\cdot\xi}$, $\xi \in \mathbb{C}^n$, is harmonic iff $0 = \Delta(e^{x\cdot\xi}) = \xi\cdot\xi\, e^{x\cdot\xi} = 0$.

For $V \neq 0$, $e^{x\cdot\zeta}$ $\zeta \in \mathbb{C}^n$ is harmonic is ~~0 = Δ(e~~ $(-\Delta + V)(e^{x\cdot\zeta}(1+w)) = 0$ on $\mathbb{R}^n$, V extended by 0. w/ $|w| \le \frac{C}{|\zeta|}$.

(*) Suppose we can do this. Choose $\zeta_1 = \xi + i(\frac{\gamma}{2} + \eta)$
$\zeta_2 = -\xi + i(\frac{\gamma}{2} - \eta)$, ξ, γ, η pairwise orthogonal. (Need $n \ge 3$ to fix γ).

$u_1 = e^{x\cdot\zeta_1} + w_1$, $u_2 = e^{x\cdot\zeta_2} + w_2$. Want to let γ fixed, ~~but send~~ ξ, η to ∞.

$$= \int (V_1 - V_2)u_1 u_2 = \int (V_1 - V_2)e^{ix\cdot\gamma} + O\left(\frac{1}{|\zeta_1|} + \frac{1}{|\zeta_2|}\right)$$ as $\xi \to \infty$.

$$= 0.$$

$\Rightarrow V_1 = V_2$ by uniqueness of FT. $L^2 \to L^2$.

For $(*)$.

$$(-\Delta + V)\left(e^{x\cdot\zeta}(1+w)\right) = -\operatorname{div}\left(\left[\zeta(1+w) + \nabla w\right] e^{x\cdot\zeta}\right) \quad (\text{formal})$$

$$= -(2\zeta\cdot\nabla w + \Delta w)e^{x\cdot\zeta} + V e^{x\cdot\zeta}(1+w) = 0.$$

$$\Leftrightarrow \quad \Delta w + 2\zeta\cdot\nabla w = V(1+w).$$

I.e., have to study $(\Delta + \zeta\cdot\nabla)w = f$

Defn. $L^2_\delta = \{u : \int (1+|x|^2)^\delta |u|^2 < \infty\}$.

Thm. If $f \in L^2_{\delta+1}$, $\exists!\, u \in L^2_\delta$ sol^n and
$$\|u\|_{L^2_\delta} \lesssim \frac{C}{|\zeta|}\|f\|_{L^2_{\delta+1}}.$$

Thm. $\Delta : L^2_\delta \to L^2_{\delta+2}$ is Fredholm for almost all δ (except for "growth rates of harmonic polynomials").

F.T.'ing $(\Delta + \zeta\cdot\nabla)w = f$ (dual variable $\zeta = \mu + i\nu$),

$$\Rightarrow \left(-|\mu|^2 + (\xi + i\eta)\cdot i\mu\right)\hat{w} = \hat{f}.$$

$$\Leftrightarrow \left((-|\mu|^2 - \eta\cdot\mu) + i(\xi\cdot\mu)\right)\hat{w} = \hat{f}.$$

Cannot divide in general!
(E.g. $\xi = e_1$, $\eta = e_2$ $\rightsquigarrow$ $|\mu|^2 + \mu_2 = 0$.

$\Leftrightarrow$ $(\mu_2 + \frac{1}{2})^2 + |\mu'|^2 = \frac{1}{4}$.

$\rightarrow \mu = (0, \mu_2, \mu')$.

I.e, codim. 2-vanishing of symbol).

Lemma Define $Z_e f = \mathcal{F}^{-1}\left(\frac{1}{\mu_e + i\mu_1}\hat{f}\right)$. Then

$$\|Z_e f\|_{L^2_\delta} \leq c\|f\|_{L^2_{\delta+1}} \quad \text{if} \quad -1 < \delta < 0.$$

Remark In $\mathbb{R}^2$, $\left(\frac{1}{\mu_1 + i\mu_2}\right)^\wedge$ is homogeneous of order $-2-(-1) = -1$.

$$F(\mu) = \int e^{in\cdot\mu} \frac{1}{n_1 + in_2}\, dn \Rightarrow \partial_{\mu_1} F = \int \frac{in_1}{\cdots} \cdots dn.$$

$$\partial_{\mu_2} F = \int \frac{in_2}{\cdots} \cdots dn.$$

$$\Rightarrow (-i\partial_{\mu_1} + \partial_{\mu_2})F = \delta.$$

$$\Rightarrow (\partial_{\mu_1} + i\partial_{\mu_2})F = i\delta.$$

$$\Rightarrow F = \frac{1}{n_e + in_2}.$$

Hence $Z_e f = \frac{1}{n_e + in_2} * f$.

Pf of Lemma   For 2 dim:

For $g \in L^2_{-\delta} = (L^2_\delta)^*$, consider

$$\langle g, z_e f \rangle_2 = \int \frac{g(u) f(v)}{(u_1 - v_1) + i(u_2 - v_2)} \, du \, dv; \text{ hence}$$

$$|\langle z_e f, g \rangle|^2 \le \left(\int \left(\int \frac{du}{(1+|u|^2)^{\delta/2} (1+|v|^2)^{\frac{1+\delta}{2}} |u-v|} \right) (1+|v|^2)^{1+\delta} |f(v)|^2 \, dv \right)$$

$$\times \left(\int \left(\int \frac{dv}{(1+|u|^2)^{\frac{\delta}{2}} (1+|v|^2)^{\frac{-(1+\delta)}{2}} |u-v|} \right) (1+|u|^2)^{-\delta} |g(u)|^2 \, du \right)$$

$$\le C \|f\|^2_{L^2_{1+\delta}} \|g\|^2_{L^2_{-\delta}} \quad (C \text{ from above integrals})$$

In n dimensions, $\int |Z f|^2 (1+|u|^2)^\delta \, du_1 \, du_2$

$$\le \int |f|^2 (1+|u|^2)^{1+\delta} \, du_1 \, du_2$$

and $(1 + u_1^2 + \cdots u_n^2)^\delta \le (1 + u_1^2 + u_2^2)^\delta$, $(1 + u_1^2 + u_2^2)^{1+\delta} \le (1 + u_1^2 + \cdots + u_n^2)^{1+\delta}$

since $-1 < \delta < 0$. Then integrate over $u_3, \ldots, u_n$. ∎

Will need another lemma: $\varphi: U \to V$. continuous choose $\Phi: f \mapsto \mathcal{F}^{-1}(\hat{f} \circ \varphi)$ is bdd on L^2.