A Gibbs point process of diffusions: existence and uniqueness

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Abstract. In this work we consider a system of infinitely many interacting diffusions as a marked Gibbs point process. With this perspective, we show, for a large class of stable and regular interactions, existence and (conjecture) uniqueness of an infinite-volume Gibbs process. In order to prove existence we use the specific entropy as a tightness tool. For the uniqueness problem, we use cluster expansion to prove a Ruelle bound, and conjecture how this would lead to the uniqueness of the Gibbs process as solution of the Kirkwood-Salsburg equation.

1 Introduction and set-up

Consider a Langevin dynamics on $\mathbb{R}^d$ of the form

$$dX_s = dB_s - \frac{1}{2} \nabla V(X_s) ds, \quad s \in [0, 2\beta], \beta > 0,$$

(1.1)
where $B$ is an $\mathbb{R}^d$-valued Brownian motion, and $V : \mathbb{R}^d \to \mathbb{R}$ is an ultracontractive potential, i.e. outside of some compact subset of $\mathbb{R}^d$,

$$\exists \delta', a_1, a_2 > 0, \quad V(x) \geq a_1 |x|^{d + \delta'} \quad \text{and} \quad \Delta V(x) - \frac{1}{2} |\nabla V(x)|^2 \leq -a_2 |x|^{2 + 2\delta'}.$$  \hspace{1cm} (1.2)

Under these conditions there exists a unique strong solution to (1.1) (see e.g. [12]), which generates an ultracontractive semigroup (see [6],[2]). Moreover, the law of $X$ starting at $X_0 = 0$ is a measure $R$ such that, for any $\delta < \delta'/2$,

$$\int e^{\|m\|_{\infty}^{d+2\delta}} R(dm) < +\infty. \hspace{1cm} (1.3)$$

For the rest of this work, let $\delta > 0$ as above be fixed.

The question we wish to explore in this work is how to construct a physically meaningful Gibbsian interaction between infinitely many such diffusions starting at random locations. More precisely, we model such a system as a marked Gibbs point process: locations and marks will describe, respectively, starting points and paths of these diffusions. We will then solve the non-trivial questions of existence and uniqueness of the infinite-volume measure for a large class of stable and regular path interactions.

After introducing the Gibbsian framework, we present an existence result via the entropy method of [11]: we use the specific entropy as a tightness tool to prove convergence of a sequence of finite-volume Gibbs measures and show that this limit satisfies the Gibbsian property (that is, the DLR equations). In section 4 we then use the method of cluster expansion – introduced by S. Poghosyan, D. Ueltschi, and H. Zessin in [8], [10] – and the Kirkwood-Salsburg equation to show a Ruelle bound for a regime of small activity, and conjecture that uniqueness of the constructed infinite-volume Gibbs process associated to path interactions follows.

## 2 Gibbsian formalism for marked point processes

The state space we consider in this work is $\mathcal{E} = \mathbb{R}^d \times C_0$, where $C_0 := C_0([0, 2\beta]; \mathbb{R}^d)$, $\beta > 0$, is the set of continuous paths $m : [0, 2\beta] \to \mathbb{R}^d$ with initial value $m(0) = 0$. An element $x = (x, m) \in \mathcal{E}$ is identified with the path $(x + m(t))_{t \in [0, 2\beta]}$ of starting point $x \in \mathbb{R}^d$ and trajectory $m \in C_0$. 
Denote by $\mathcal{M}$ the set of locally-finite point measures (or configurations) on $\mathcal{E}$, which are of the form $\gamma = \sum_i \delta_{(x_i,m_i)} \in \mathcal{M}$; we often identify a configuration $\gamma$ with its support $\{(x_i,m_i)\}_i \subset \mathcal{E}$.

Let $\mathcal{B}_b(\mathbb{R}^d)$ be the subset of bounded Borel sets of $\mathbb{R}^d$. Let $\mathcal{M}_f$ denote the subset of finite configurations, and for any $\Lambda \in \mathcal{B}_b(\mathbb{R}^d)$, let $\mathcal{M}_\Lambda \subset \mathcal{M}_f$ denote the restriction to starting points inside $\Lambda$, and for any configuration $\gamma \in \mathcal{M}$, let $\gamma_\Lambda := \gamma \cap (\Lambda \times C_0) \in \mathcal{M}_\Lambda$.

Let $\mathcal{P}(\mathcal{M})$ denote the set of probability measures on $\mathcal{M}$: these are called marked point processes. As reference process we consider, for any $\Lambda \in \mathcal{B}_b(\mathbb{R}^d)$, the marked Poisson point process $\pi_\Lambda$ on $\mathcal{E}$ with intensity measure $z \, dx_\Lambda \otimes R(dm)$. The coefficient $z$ is a positive real number, $dx_\Lambda$ is the Lebesgue measure on $\Lambda$, and the probability measure $R$ is the path measure of the solution of (1.1) starting at 0. In other words, the starting points are drawn in $\Lambda$ according to a Poisson process, and the marks are diffusion paths starting at these Poisson points.

We add interaction between the points of a configuration by considering an energy functional that takes into account both the locations and the marks.

**Assumption 1.1** For any finite marked point configuration $\gamma = \{x_1, \ldots, x_N\} \in \mathcal{M}_f, N \geq 1$, its energy is given by the following functional

$$H(\gamma) = \sum_{i=1}^N \Psi(x_i) + \sum_{i=1}^N \sum_{j<i} \Phi(x_i, x_j) \in \mathbb{R} \cup \{+\infty\},$$

where

\begin{itemize}
  \item The \textbf{self-potential} term $\Psi$ satisfies $\inf_{x \in \mathbb{R}^d} \Psi(x,m) \geq -k_\Psi \|m\|^{d+\delta}$ for some constant $k_\Psi > 0$;
  \item The \textbf{two-body potential} $\Phi$ is defined by
    $$\Phi(x_i, x_j) = \left(\phi(x_i - x_j) + \int_0^{2\beta} \tilde{\phi}(m_i(s) - m_j(s)) \, ds\right)\mathbf{1}_{\{|x_i - x_j| \leq a_0 + \|m_i\| + \|m_j\|\}},$$
\end{itemize}

where $\phi$ (acting on on the initial location of the diffusions) is a \textit{radial} (i.e. $\phi(x) = \phi(|x|)$) and \textit{stable} $\mathbb{R}$-valued pair potential in the sense of [13], with stability constant $c_\phi \geq 0$, bounded from below, with $\phi(u) \leq 0$ for $u \geq a_0$ (see Figure 1.1); $\tilde{\phi}$ (acting on the dynamics of the diffusions) is a non-negative pair potential.
Figure 1.1: An example of radial and stable pair potential $\phi$ is the Lennard-Jones potential $\phi(u) = 16\left( \frac{3}{2} \right)^{12} - \left( \frac{3}{2} \right)^{6}$; its zero is at $a_0 = \frac{3}{2}$.

Remark 1.2  
(i) The stability of the point-interaction potential $\phi$ and the non-negativity of the mark-interaction potential $\tilde{\phi}$ guarantee stability (in the sense introduced in Lemma 1.5) of the energy $H$ of a marked-point configuration; the fact that $\phi$ is bounded from below is used to prove the stability of the conditional energy (see Lemma 1.7).

(ii) The indicator function in (1.5) can be interpreted as follows: when the starting points are far enough from each other, the two diffusions do not interact; if their paths do not intersect, they may interact only if $|x_1 - x_2| \leq a_0 + \|m_1\|_\infty + \|m_2\|_\infty$. See Figure 1.2. Notice that the range of interaction is finite but not uniformly bounded.

Definition 1.3 For any $\Lambda \in \mathcal{B}(\mathbb{R}^d)$, the free-boundary-condition finite-volume Gibbs measure on $\Lambda$ with energy $H$ and activity $z > 0$ is the probability measure $P_{\Lambda}^z$ on $\mathcal{M}_\Lambda$.
defined by
\[ P_\Lambda^z(d\gamma) := \frac{1}{Z_\Lambda^z} e^{-H(\gamma_\Lambda)} \pi_\Lambda^z(d\gamma). \tag{1.6} \]

In this work we investigate the existence and uniqueness, as \( \Lambda \) increases to cover the whole space \( \mathbb{R}^d \), of an infinite-volume Gibbs measure, in the following sense:

**Definition 1.4** A probability measure \( P \) on \( \mathcal{M} \) is said to be an infinite-volume Gibbs measure with energy \( H \) and activity \( z > 0 \), denoted by \( P_{\infty}^{G(H,z)} \), if it satisfies, for any \( \Lambda \in \mathcal{B}(\mathbb{R}^d) \) and any positive, bounded, and measurable functional \( F : \mathcal{M} \rightarrow \mathbb{R} \), the following DLR equation (for Dobrushin-Landford-Ruelle)
\[
\int_{\mathcal{M}} F(\gamma) P(d\gamma) = \int_{\mathcal{M}} \frac{1}{Z_\Lambda(\xi)} \int_{\mathcal{M}_\Lambda} F(\gamma_\Lambda(\xi_{\Lambda^c})) e^{-H_\Lambda(\gamma_\Lambda(\xi_{\Lambda^c}))} \pi_\Lambda^z(d\gamma) P(d\xi), \tag{DLR}
\]
where \( H_\Lambda(\gamma) \) is the conditional energy of the configuration \( \gamma \) in \( \Lambda \) given its exterior:
\[
H_\Lambda(\gamma) := \lim_{r \to +\infty} H(\gamma_{\Lambda \oplus B(0,r)}) - H(\gamma_{\Lambda \oplus B(0,r) \setminus \Lambda}), \tag{1.7}
\]
with \( \Lambda \oplus B(0,r) := \{ x \in \mathbb{R}^d : \exists y \in \Lambda, \ |y - x| \leq r \} \).

### 3 Existence of an infinite-volume Gibbs point process via the entropy method

Under Assumption 1.1 on the energy functional \( H \), the following three lemmas provide the groundwork for the existence theorem.

**Lemma 1.5** The following stability condition holds: setting \( \mathcal{C}_H := k \psi \vee \mathcal{C}_0 \),
\[
H(\gamma) \geq -\mathcal{C}_H \sum_{(x,m) \in \gamma} \left( 1 + \|m\|^{d+\delta}_\infty \right), \quad \gamma \in \mathcal{M}_f. \tag{1.8}
\]

In order to control the support of the Gibbs point process, we define the subset of tempered configurations as the union \( \mathcal{M}^{\text{temp}} := \bigcup_{t \in \mathbb{N}} \mathcal{M}^t \), where \( \mathcal{M}^t \) is the set of all configurations \( \gamma \in \mathcal{M} \) such that, for all \( l \in \mathbb{N}^* \), \( \sum_{(x,m) \in \gamma_{B(0,l)}} (1 + \|m\|^{d+\delta}_\infty) \leq t^d \).
Lemma 1.6 For any bounded $\Lambda \subset \mathbb{R}^d$ and $t \geq 1$, there exists a random variable $r = r(\gamma, t) < +\infty$ such that the limit in (1.7) stabilises, i.e.

$$H_\Lambda(\gamma) = H(\gamma_{\Lambda \cap B(0, r)}) - H(\gamma_{\Lambda \cap B(0, r) \setminus \Lambda}).$$

We say that $r(\gamma, t)$ is the finite but random range of the interaction $H_\Lambda(\gamma)$.

Lemma 1.7 Fix $\Lambda \in \mathcal{B}_b(\mathbb{R}^d)$. For any $t \geq 1$, there exists a constant $\xi' (\Lambda, t) \geq 0$ such that the following stability of the conditional energy holds: uniformly for all $\xi \in \mathcal{M}^t$,

$$H_\Lambda(\gamma, \xi_{\Lambda \cap \cdot}) \geq -\xi'(\Lambda, t) \sum_{(x,m) \in \gamma_\Lambda} (1 + \|m\|_{\infty}^{d+\delta}), \quad \gamma \in \mathcal{M}_\Lambda. \quad (1.9)$$

We endow the set $\mathcal{P}(\mathcal{M})$ of probability measures on $\mathcal{M}$ with the topology of local convergence (see [4], [5]). More precisely,

Definition 1.8 A functional $F$ on $\mathcal{M}$ is called local and tame if there exist a set $\Delta \in \mathcal{B}_b(\mathbb{R}^d)$ and a constant $a > 0$ such that, for all $\gamma \in \mathcal{M}$, $F(\gamma) = F(\gamma_{\Delta})$ and $|F(\gamma)| \leq a \left(1 + \sum_{(x,m) \in \gamma_\Delta} (1 + \|m\|_{\infty}^{d+\delta})\right)$.

We denote by $\mathcal{L}$ the set of all local and tame functionals. The topology $\tau_\mathcal{L}$ of local convergence on $\mathcal{P}(\mathcal{M})$ is defined as the weak* topology induced by $\mathcal{L}$, i.e. the smallest topology on $\mathcal{P}(\mathcal{M})$ under which all the mappings $P \rightarrow \int F dP$, $F \in \mathcal{L}$, are continuous.

Let us now recall the concept of specific entropy of a probability measure on $\mathcal{M}$.

Definition 1.9 Given two probability measures $Q$ and $Q'$ on $\mathcal{M}$, the specific entropy of $Q$ with respect to $Q'$ is defined by

$$I(Q|Q') = \lim_{\Lambda_n \nearrow \mathbb{N}^d} \frac{1}{|\Lambda_n|} I_{\Lambda_n}(Q|Q'),$$

where $\Lambda_n = [-n, n)^d$, and the relative entropy of $Q$ with respect to $Q'$ on $\Lambda$ is defined as

$$I_\Lambda(Q|Q') = \begin{cases} \int \log f \, dQ_\Lambda & \text{if } Q_\Lambda \ll Q'_\Lambda \text{ with } f := \frac{dQ_\Lambda}{dQ'_\Lambda}, \\ +\infty & \text{otherwise}, \end{cases}$$

where $Q_\Lambda$ (resp. $Q'_\Lambda$) is the image of $Q$ (resp. $Q'$) under the mapping $\gamma \mapsto \gamma_\Lambda$. 

A. Zass: Interacting diffusions
The specific entropy with respect to $\pi^z$ is well defined as soon as $Q$ is invariant under translations on the lattice. Moreover, we underline that for any $a > 0$, the $a$-entropy level set

$$\mathcal{P}(\mathcal{M}) \leq a := \{ Q \in \mathcal{P}(\mathcal{M}) : I(Q | \pi^z) \leq a \}$$

is relatively compact for the local convergence topology $\tau_{\mathcal{S}}$, as proved in [5].

Putting together the technical conditions described in this section yields the existence of an infinite-volume Gibbs measure $P^z$, for any activity $z > 0$.

**Theorem 1.10** For any energy functional $H$ as in Assumption 1.1 and any activity $z > 0$, there exists at least one infinite-volume Gibbs measure $P^z \in \mathcal{G}(H, z)$.

**Sketch of proof.**

(i) For $\Lambda_n = [-n, n)^d$, consider the sequence $(P^z_n)_{n \geq 1}$ of finite-volume Gibbs measures, and build the empirical field $(\bar{P}^z_n)_{n \geq 1}$ by stationarising it w.r.t. lattice translations.

(ii) Use uniform bounds on the specific entropy to show the convergence, up to a subsequence, to an infinite-volume measure $P^z$.

(iii) Prove, using an ergodic property, that $P^z$ carries only the space of tempered configurations.

(iv) Noticing that $\bar{P}^z_n$ does not satisfy the (DLR) equations, introduce a new sequence $(\tilde{P}^z_n)_{n \geq 1}$ asymptotically equivalent to $(\bar{P}^z_n)_{n \geq 1}$ but satisfying (DLR).

(v) Use appropriate approximation technique to show that $P^z$ satisfies (DLR) too.

For details, see [11].

**Example 1.11** Let $d = 2$. A concrete example of functions satisfying the above assumptions is as follows:

Consider as reference diffusion a Langevin dynamics with $V(x) = |x|^4$; the diffusion is ultracontractive with $\delta' = 2$. The invariant measure $\mu(dx) = e^{|x|^4} dx$ is a Subbotin measure (see [15]).

Consider as self interaction $\Psi(x) = -||m||_{\infty}^{3/2}$; as interaction between the initial locations a *Lennard-Jones* pair potential $\phi(u) = au^{-12} - bu^{-6}$, $a, b > 0$; as interaction between the marks any non-negative pair potential $\tilde{\phi}$. 
4 Uniqueness of Gibbs measure via cluster expansion

The method of cluster expansion relies in finding a regime of small activity \(0 < z \leq \bar{z}\) in which the partition function \(Z^z_\Lambda\) can be written as the exponential of an absolutely converging series of cluster terms. It should then be possible to write an equation (the so-called Kirkwood-Salsburg equation, see e.g. [14]) for the correlation functions of the infinite-volume Gibbs measure \(P^z\) constructed above. We conjecture that under some assumptions, such an equation has a unique solution, which would lead to the uniqueness of the infinite-volume Gibbs measure. Here we use a strategy developed in [9]. For this section, we make the following additional

**Assumption 1.12** The potential \(\phi\) (on initial locations of the diffusions) is integrable in \(\mathbb{R}^d\): \(\|\phi\|_1 < +\infty\); the potential \(\tilde{\phi}\) (on the dynamics of the diffusions) is bounded: \(\|\tilde{\phi}\|_\infty < +\infty\).

The partition function is given, for any \(L \subset \mathbb{R}^d\), by

\[
Z^z_\Lambda = 1 + \sum_{N \geq 1} \frac{z^N}{N!} \int_{(\Lambda \times C_0)^N} \exp \left\{ - \sum_{1 \leq i \leq N} \Psi(x_i, m_i) - \sum_{1 \leq i < j \leq N} \phi(|x_i - x_j|) \right\} \\
+ \int_0^{2\beta} \tilde{\phi}(|m_i(s) - m_j(s)|) \, ds \begin{cases} 1 & \text{if } |x_i - x_j| \leq a_0 + \|m_i\|_\infty + \|m_j\|_\infty \end{cases},
\]

\(dx_1 \cdot \cdot \cdot dx_N \, R(dm_1) \cdot \cdot \cdot R(dm_N).\) \hspace{1cm} (1.10)

**Theorem 1.13** Consider an energy functional \(H\) satisfying Assumption 1.1 and Assumption 1.12. Then the two-body potential \(\Phi\) satisfies a modified regularity condition. Therefore, there exists \(\bar{z} > 0\) such that, for any activity \(z \leq \bar{z}\), the partition function above converges absolutely and a Ruelle bound holds.

**Proof.** In order to guarantee the absolute convergence of (1.10), we check whether the pair potential \(\Phi\) satisfies a modified c-regularity for the functional \(\alpha\) (terminology from [10]; introduced in [8]), i.e. that for any \(x_1 = (x_1, m_1)\), the following inequality holds

\[
z^c \int e^{\alpha(x_2)} |\Phi(x_1, x_2)| e^{-\Psi(x_2)} \, dx_2 \, R(dm_2) \leq \alpha(x_1).
\]

We consider here \(c = c_\phi\), and a function of the form \(\alpha(x, m) = \alpha(m) = a_1 \|m\|_\infty^d\), where

\[
a_1 = \|\phi\|_1 + \left(2\beta \|\tilde{\phi}\|_\infty k_d b_d (a_0^d + 1)\right),
\]
with $k_d$ such that $(x + y + z)^d \leq k_d(x^d + y^d + z^d)$, and $b_d$ the volume of the unit ball in $\mathbb{R}^d$. Recalling that the self potential $\Psi$ is such that $\Psi(x) \geq -k\varphi\|m\|_{\infty}^{d+\delta}$. Set $\rho := \int e^{(a_1 + k\varphi)\|m\|_{\infty}^d} R(dm) < +\infty$; the modified regularity condition reads

$$Ze^{\rho} \int_{C_0} e^{a_1\|m_2\|_{\infty}^d} \int_{\mathbb{R}^d} |\phi(x_2 - x_1)| + \left( \int_0^{2\beta} \phi'(m_2(s) - m_1(s))ds \right) \mathbb{1}_{\{|x_1 - x_2| \leq a_0 + \|m_1\|_{\infty} + \|m_2\|_{\infty}} \right) dx_2 e^{k\varphi\|m_2\|_{\infty}^d + \delta} R(dm_2) \leq a_1\|m_1\|_{\infty}^d.$$ 

Estimating the l.h.s. leads to the following condition:

$$z \leq \frac{\|m_1\|_{\infty}^d}{\rho e^{\rho} (\|m_1\|_{\infty}^d + 1)},$$

which holds as soon as $z \leq (2\rho e^{\rho})^{-1} =: \bar{z} = \inf_{m_1} \frac{\|m_1\|_{\infty}^d}{\rho e^{\rho} (\|m_1\|_{\infty}^d + 1)}$. Applying results in [8], this implies the absolute convergence of (1.10). Moreover, in [9] S. Poghosyan and H. Zessin prove that a Ruelle bound also holds.

The unique step towards uniqueness which is now missing is the proof that the Kirkwood-Salsburg equation has a unique solution. We state the following conjecture:

**Conjecture 1.14** For any activity $z \leq \bar{z}$, the Kirkwood-Salsburg equation has a unique solution.

Assuming the above conjecture to hold true, we obtain the following

**Corollary 1.15** For any activity $z \leq \bar{z}$, the infinite-volume measure $P^z$ constructed in Theorem 1.10 is the unique Gibbs measure in $\mathcal{G}(H, z)$.

**Conclusions and outlook.** In [3], D. Dereudre showed the equivalence between the law of an infinite-dimensional interacting SDE with Gibbsian initial law, and a Gibbs point process on the path space, with a certain energy functional.

It is a natural question to ask whether a Gibbs point process with energy functional $H$ as in Assumption 1.1 is the law of infinite dimensional interacting SDE. Using Malliavin derivatives, D. Dereudre proved that Gibbs point processes with regular $H$ are the law of SDEs with a certain non-markovian drift. See [1] and [7] in the lattice case.

The existence and uniqueness results presented here could therefore be useful to obtain a criterium for the solution of infinite-dimensional SDEs. This is a work in progress.
Bibliography


