Limit behaviour of random walks with elastic screens

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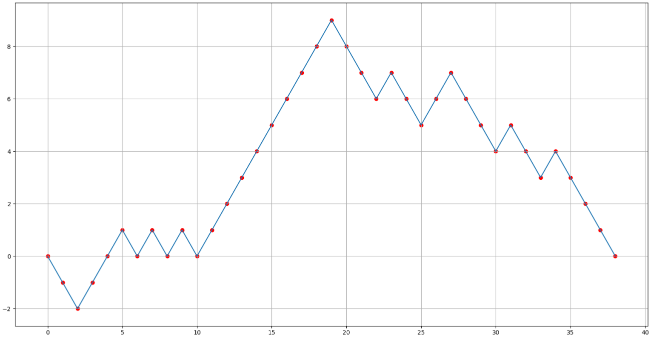
Let $\{S(n)\}$ be a simple random walk, i.e.

$$S(n) = \sum_{i=1}^{n} \xi_i,$$

where $\{\xi_i\}$ are independent ± 1 with probability $\frac{1}{2}$.
Extend $S(n)$ for all t by linearity:

$$S(t) = S(n) + (t-n)(S(n+1) - S(n)), \ t \in [n, n+1].$$



Consider a random walk $\{\tilde{S}(n)\}$, which behaves as $\{S(n)\}$ everywhere except 0. In 0 it stops for some random amount of time and then continues its way as $\{S(n)\}$.

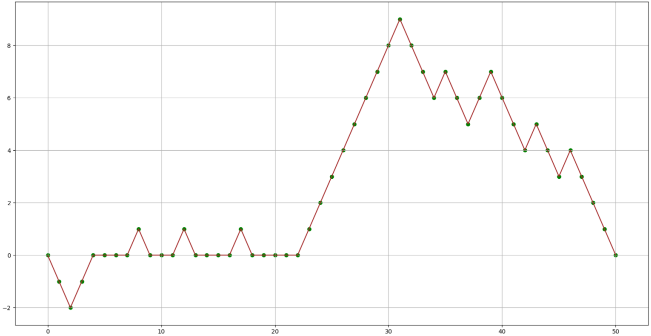
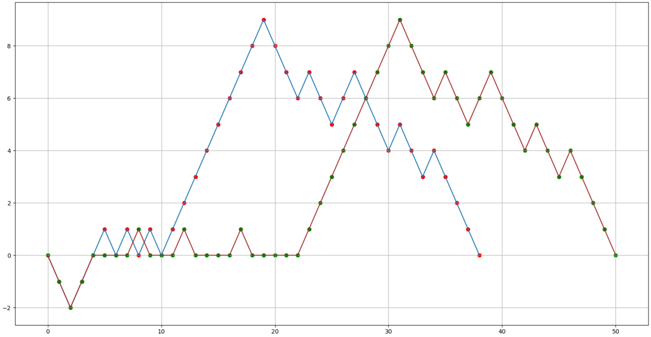


Figure: $\tilde{S}(n)$

Two random walks together.



The Donsker Theorem

In $\mathrm{C}([0,T])$

$$\frac{S(nt)}{\sqrt{n}} \xrightarrow{w} W(t), n \to \infty. \tag{1}$$

Our goal is to analyse the behaviour of a limiting random process for:

$$\tilde{X}_n(t) = \frac{\tilde{S}(nt)}{\sqrt{n}}.$$

Let times of lagging in zero be a sequence of i.i.d.r.v. $\{\eta_n\}_{n=1}^{\infty}$ and

$$\alpha(t) = t + \sum_{i=1}^{\tau_0(nt)} \eta_i, \ t \geq 0, \tag{2}$$

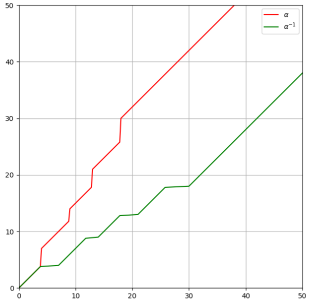
where $\tau_0(t) = \#\{k : S(k) = 0, 0 < k \leq t\}$, i.e. number of returns to zero before the time t.

Consider

$$\alpha^{-1}(t) = \inf\{x : \alpha(x) \geq t\}, \ t \geq 0. \tag{3}$$

Lemma

$$\tilde{S}(t) = S(\alpha^{-1}(t)) \tag{4}$$



Theorem 1.

Let $\eta_i \geq 0$ be i.i.d.r.v. with $\mathbb{E}\eta_i < \infty$, $i \geq 1$. Then the sequence $\tilde{X}_n(t) = \frac{\tilde{S}(nt)}{\sqrt{n}}, n \geq 1$ converges weakly in $\mathrm{C}([0,T])$ to a Wiener's process $W(t)$:

$$\tilde{X}_n(t) \xrightarrow{w} W(t),\ n \to \infty. \tag{5}$$

Let

$$h_n(t) = \frac{\alpha^{-1}(nt)}{n},$$

then

$$\tilde{X}_n(t) = \frac{\tilde{S}(nt)}{\sqrt{n}} = \frac{S(\alpha^{-1}(nt))}{\sqrt{n}} = \frac{S(n\frac{\alpha^{-1}(nt))}{n})}{\sqrt{n}} = X_n(h_n(t)). \tag{6}$$

Recall that

$$h_n(t) = \frac{\alpha^{-1}(nt)}{n}, \quad \frac{\alpha(nt)}{n} = t + \frac{1}{n}\sum_{i=1}^{\tau_0(nt)} \eta_i, \quad \tilde{X}_n(t) = X_n(h_n(t)).$$

We will proceed by the following steps:

1. We will show that

$$\forall T > 0 \ h_n(t) \overset{[0,T]}{\rightrightarrows} t, n \to \infty, \text{a.s.}$$

   by means of showing:

   1. $\frac{\alpha(n)}{n} \to 1, n \to \infty$, a.s.
   2. $\forall T > 0 \ \frac{\alpha(nt)}{n} \overset{[0,T]}{\rightrightarrows} t, n \to \infty$, a.s.

2. Use Skorohod's representation theorem for the pair (X_n, h_n) to obtain a.s. convergence of copies, that are subjected to the same distributions:

$$X_n(t) \underset{n\to\infty}{\rightrightarrows} W(t) \text{ a.s.}, \quad h_n(t) \underset{n\to\infty}{\rightrightarrows} t \text{ a.s.}.$$

3. Derive the claim on the new probability space and thus on the original one.

Consider the Markov chain $S^p(n)$ which if it is in state 0 has probability p of leaving zero during the current step. That is

$$S^p(n) = \begin{cases} \begin{cases} S^p_{n-1} \begin{matrix} +1, \text{w. pr. } 0.5 \\ -1, \text{w. pr. } 0.5 \end{matrix} \end{cases} & \text{if } S^p_{n-1} \neq 0 \\ \begin{cases} +1, \text{w. pr. } 0.5 \\ -1, \text{w. pr. } 0.5 \end{cases} & \text{w. pr. } p \text{ if } S^p(n-1) = 0, \\ 0, & \text{w. pr. } 1-p \text{ if } S^p(n-1) = 0, \end{cases} \tag{7}$$

$S^p(n)$ is equivalent to $\tilde{S}(n)$ with η_i geometrically distributed with parameter p. So in $\mathrm{C}([0,T])$

$$\frac{S^p(nt)}{\sqrt{n}} \xrightarrow{w} W(t), n \to \infty.$$

Lagging depending on n

Let $S^p(n)$ be a Markov chain on $\mathbb{Z}$ with

$$p_{0,0} = 1 - p,\ \ p_{0,\pm 1} = \frac{p}{2},$$
$$\forall i \neq 0\ p_{i,i\pm 1} = \frac{1}{2}.$$

And let $p = p_n = \frac{\rho}{n^\gamma}$. Consider

$$X_n^{p_n}(t) = \frac{S^{p_n}(nt)}{\sqrt{n}}.$$

Theorem 2

In $\mathrm{C}([0,T])$:

$$\text{if } \gamma < 0.5,\ \text{ then } X_n^{p_n}(t) \xrightarrow[n\to\infty]{w} W(t),$$
$$\text{if } \gamma > 0.5,\ \text{ then } X_n^{p_n}(t) \xrightarrow[n\to\infty]{w} 0,$$
$$\text{if } \gamma = 0.5,\ \text{ then } X_n^{p_n}(t) \xrightarrow[n\to\infty]{w} W_{\text{sticky}}(t).$$

As we have seen in example, $S^{p_n}(n) \overset{d}{=} \tilde{S}(n)$ with $\eta_i^{(n)} \sim \mathrm{Geom}(p_n)$. So we will prove the theorem for such $\tilde{S}(n)$.

Now $\alpha(t) = \alpha_n(t)$ depends on n. Still we write

$$X_n^{p_n}(t) = X_n(h_n(t)), \quad h_n(t) = \frac{\alpha_n^{-1}(nt)}{n} \tag{8}$$

and

$$\frac{\alpha_n(nt)}{n} = t + \frac{1}{n} \sum_{i=1}^{\tau_0(nt)} \eta_i^{(n)} = t + \frac{n^\gamma}{\sqrt{n}} \frac{1}{\sqrt{n}} \sum_{i=1}^{\tau_0(nt)} \frac{\eta_i^{(n)}}{n^\gamma}. \tag{9}$$

Theorem

Let $W(t)$ — be a Brownian motion in $\mathbb{R}$ and $L_W^0(t)$ — be its local time at 0, i.e

$$L_W^0(t) = \lim_{\epsilon \to 0} \frac{1}{2\epsilon} \int_0^t 1_{|W(s)| \le \epsilon} ds.$$

Then in $\mathrm{C}([0, T])$:

$$\left(\frac{\tau_0(nt)}{\sqrt{n}}, \frac{S(nt)}{\sqrt{n}} \right) \xrightarrow{w} (L_W^0(t), \ W(t)), \ n \to \infty.$$

Main steps of the proof

We will proceed by the following steps:

1. Use Skorohod's representation theorem to the pair $(\tau_0(t), S(t))$.
2. On a new probability space prove convergence of

$$\forall T>0 \quad \sup_{t\in[0,T]}\left|\frac{1}{\sqrt{n}}\sum_{i=1}^{\sqrt{n}L_W^0(t)}\frac{\eta_i^{(n)}}{n^{\gamma}}-\frac{L_W^0(t)}{\rho}\right|\xrightarrow{\mathbb{P}}0, n\to\infty, \tag{10}$$

1. $$\frac{1}{\sqrt{n}}\sum_{i=1}^{\sqrt{n}t}\frac{\eta_i^{(n)}}{n^{\gamma}}\xrightarrow{\mathbb{P}}\frac{t}{\rho}, n\to\infty.$$
2. $$\forall T>0 \quad \sup_{t\in[0,T]}\left|\frac{1}{\sqrt{n}}\sum_{i=1}^{\sqrt{n}t}\frac{\eta_i^{(n)}}{n^{\gamma}}-\frac{t}{\rho}\right|\xrightarrow{\mathbb{P}}0, n\to\infty,$$

Recall that

$$\tilde{X}_n(t) = X_n(h_n(t)), \quad \frac{\alpha_n(nt)}{n} = t + \frac{n^\gamma}{\sqrt{n}} \frac{1}{\sqrt{n}} \sum_{i=1}^{\tau_0(nt)} \frac{\eta_i^{(n)}}{n^\gamma}.$$

1. For $\gamma < 0.5$:

$$\frac{\alpha_n(nt)}{n} \xrightarrow[n\to\infty]{w} t. \tag{11}$$

2. Invoke Skorohod's theorem once again for the triplet of random elements

$$(\tau_0(t), S(t), \alpha_n(t)).$$

3. Appeal to the proof of theorem 1.

$$X_n^{p_n}(t) \xrightarrow[n\to\infty]{w} W(t).$$

Recall that

$$\tilde{X}_n(t) = X_n(h_n(t)), \quad \frac{\alpha_n(nt)}{n} = t + \frac{n^\gamma}{\sqrt{n}} \frac{1}{\sqrt{n}} \sum_{i=1}^{\tau_0(nt)} \frac{\eta_i^{(n)}}{n^\gamma}.$$

1. For $\gamma > 0.5$:

for every $\delta > 0$ on $[\delta, T]$

$$\frac{\alpha_n(nt)}{n} \underset{n\to\infty}{\overset{w}{\to}} \infty. \tag{12}$$

2. Invoke Skorohod's theorem once again for the triplet of random elements

$$(\tau_0(t), S(t), \alpha_n(t)).$$

3. Show that

$$h_n(t) \rightrightarrows 0,$$

4. Hence the claim on the new probability space.
5. Hence on the original one.

$$X_n^{p_n}(t) \underset{n\to\infty}{\overset{w}{\longrightarrow}} 0.$$

Recall that

$$\frac{\alpha_n(nt)}{n} = t + \frac{n^\gamma}{\sqrt{n}} \frac{1}{\sqrt{n}} \sum_{i=1}^{\tau_0(nt)} \frac{\eta_i^{(n)}}{n^\gamma}.$$

1. For $\gamma = 0.5$:

$$\frac{\alpha(nt)}{n} \underset{n\to\infty}{\overset{w}{\to}} t + L^0_W(t)/\rho. \tag{13}$$

2. As $\frac{\alpha(nt)}{n}$ is strictly monotonous, its generalized inverse is continious.
3. Hence

$$\frac{\alpha^{-1}(nx)}{n} = Inv\left[\frac{\alpha(nt)}{n}\right](x) \overset{w}{\to} Inv[t + L^0_W(t)/\rho](x). \tag{14}$$

4. Set

$$W_{\mathsf{sticky}}(x) = W(Inv[t + L^0_W(t)/\rho](x))$$

Thank you for your attention!