Wave equation driven by general stochastic measure

Iryna Bodnarchuk

Based on the joint research with Vadym Radchenko

Taras Shevchenko National University of Kyiv, Ukraine

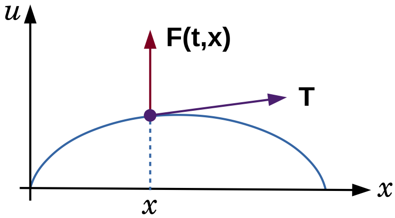
Potsdam, April 1-3, 2019

Consider the next equation

$$\frac{\partial^2 u(t,x)}{\partial t^2} = a^2 \Delta u(t,x) + f(t,x). \qquad (1)$$

This equation describes some waves.

1. Let we have a homogeneous string.



Here $a = \sqrt{\frac{T}{\rho}}$, T is a string tension, ρ is a string density, $f = \frac{F(t,x)}{\rho}$, where $F(t,x)$ is an external force. $u(t,x)$ describes displacement of the point x.

2. Electrical oscillations in wires.
Concider the wire with electric current. Suppose that losses due to isolation are absent and resistance is very small.

$$\frac{\partial^2 I(t,x)}{\partial t^2} = a^2 \frac{\partial^2 I(t,x)}{\partial x^2}.$$

Here $a = \sqrt{\frac{1}{LC}}$, L is a coefficient of self-induction, C is a coefficient of capacity.
$I(t,x)$ is an amperage at time t in point x.

3. Sound propagation in gas.

The next equation describes sound propagation

$$\frac{\partial^2 \rho(t,x)}{\partial t^2} = a^2 \Delta \rho(t,x),$$

were $a = \sqrt{\frac{C_p\, p_0}{C_v\, \rho_0}}$ is velosity of sound,
C_p is a heat capacity with constant pressure,
C_v is a heat capacity with constant volume,
p_0 is an initial pressure,
ρ_0 is an initial density.
$\rho(t,x)$ is an density of gas at time t in point x.

3. The internal structure of the sun (R.C. Dalang, 2009).
The motion of the sun's surface is investigated for obtaining information about the internal structure of the sun.

The sun's surface moves in a complex manner: at any given time, any point on the surface is typically moving towards or away from the center. There are also waves going around the surface, as well as shock waves propagating through the sun itself, which cause the surface to pulsate.

A question of interest to solar geophysicists is to determine the origin of these shock waves. One school of thought is that they are due to turbulence, but the location and intensities of the shocks are unknown, so a probabilistic model can be considered.

A model that was proposed by P. Stark of U.C. Berkeley is that the main source of shocks is located in a spherical zone inside the sun, which is assumed to be a ball of radius R. Assuming that the shocks are randomly located on this sphere, the equation for the dilatation throughout the sun (J.L. Davis, 1988) would be

$$\frac{\partial^2 u(t,x)}{\partial t^2} = a^2(x)\rho(x)\left(\nabla\left(\frac{1}{\rho(x)}\nabla u(t,x)\right) + \nabla F(t,x)\right),$$

were $x \in B(0,R)$, $\rho(x)$ is an density at x, $a(x)$ is the speed of wave propagation at x, vector $F(t,x)$ models the shock that originates at time t and position x. F can be represented as 3-dimensional Gaussian noise.

Stochastic measure

Let $L_0(\Omega, \mathcal{F}, P)$ be the set of all (equivalence classes of) real-valued random variables defined on $(\Omega, \mathcal{F}, P)$, X be an arbitrary set and $\mathcal{B}(X)$ a σ-algebra of Borel subsets of X.

Definition

Any σ-additive mapping $\mu : \mathcal{B}(X) \to L_0(\Omega, \mathcal{F}, P)$ is called a *stochastic measure* (SM).

In S. Kwapień, W.A. Woyczyński, 1992 such μ is called a general stochastic measure. This underlines the fact that we do not require the fulfillment of any other assumptions except σ-additivity.

Examples of stochastic measures

- Let $X(t)$, $t \in [0, T]$ be a stochastic process with independent increments, $X(0) = 0$. Then the additive set function
$$m((s,t]) = X(t) - X(s)$$
can be extended to a SM on $\mathcal{B}([0,T])$ if and only if

$$\sup_{0=t_0<t_1<\ldots<t_{n-1}<t_n=T} \sum_{i=1}^{n} \left| \mathsf{E}[\![X(t_i) - X(t_{i-1})]\!] \right| < \infty\,,$$

where

$$[\![x]\!] = \begin{cases} x, & |x| \le 1, \\ x/|x|, & |x| > 1\,. \end{cases}$$

Examples of stochastic measures

∘ $\mu(A) = \int_0^T 1\!\!\mathrm{I}_A(t)dX(t), A \in \mathcal{B}([0,T])$, where $X(t), t \in [0,T]$ is a continuous square integrable martingale or a fractional Brownian motion with Hurst index $H > 1/2$.

Theorem 1.1 (T. Memin, Yu. Mishura, E. Valkeila, 2001) Let $f \in L_{\frac{1}{H}}([a,b])$ and $f = 0$ outside [a,b]. $X(t)$ be a fBm with $H > 1/2$. Then there exists a constsnt $c(H,r)$ such that for every $r > 0$ and for every a, b , $0 \le a < b < \infty$ we have

$$\mathsf{E}\left(\left|\int_a^b f(t)dX(t)\right|^r\right) \le c(H,r)\|f\|^r_{L_{\frac{1}{H}}([a,b])}.$$

∘ Let (ξ_n) be a sequence of random variables such that $\sum_{n\geq 1}\xi_n$ converges unconditionally in probability, and let m_n be a charge on $\mathcal{B}(\mathsf{X})$ such that $\forall\, A\in\mathcal{B}(\mathsf{X}):\ |\mathsf{m}_n(A)|\leq 1$. Then

$$\mu(A)=\sum_{n\geq 1}\xi_n\mathsf{m}_n(A)$$

is a SM on $\mathcal{B}(\mathsf{X})$.

Theorem A.1.1 (S. Kwapień, W.A. Woyczyński, 1992) Let (ξ_n) be a sequence of elements of $\mathsf{L}_0(\mathsf{F})$, where F is a Banach space. Then the series $\sum_{n\geq 1}\xi_n$ converges unconditionally in $\mathsf{L}_0(\mathsf{F})$ if and only if for each bounded sequence $\lambda_1,\lambda_2,\ldots\in\mathbb{R}$ the series converges in $\mathsf{L}_0(\mathsf{F})$.

Theorem 8.6 (L. Drewnovski, 1972) Let μ_n be SMs on $\mathcal{B}(\mathsf{X}), n\geq 1$ and $\forall\, A\in\mathcal{B}(\mathsf{X})\ \ \exists\,\mu(A)=\mathsf{P}\lim_{\mathsf{n}\to\infty}\mu_{\mathsf{n}}(\mathsf{A})$. Then μ is a SM on $\mathcal{B}(\mathsf{X})$.

Stochastic integral

For deterministic measurable functions $g : \mathsf{X} \to \mathbb{R}$ an integral of the form

$$\int_{\mathsf{X}} g \, d\mu$$

is defined and has a standart construction with approximation by simple functions.

In particular, every bounded measurable g is integrable with respect to any μ.

Wave equation with stochastic measure

Consider the Cauchy problem for the stochastic wave equation

$$\begin{cases} \dfrac{\partial^2 u(t,x)}{\partial t^2} = a^2 \Delta_x u(t,x) + f(t,x,u(t,x)) + \sigma(t,x)\dot{\mu}(t), \\ u(0,x) = u_0(x); \quad \dfrac{\partial u(0,x)}{\partial t} = v_0(x), \end{cases} \tag{2}$$

where $(t,x) \in [0,T] \times \mathbb{R}^d$, $d = 1,2,3$, $a > 0$, Δ_x is Laplace operator, μ is a stochastic measure defined on $\mathcal{B}([0,T])$.

We investigate the mild solution of (2), i. e., any measurable random function $u(t,x) = u(t,x,\omega) : [0,T] \times \mathbb{R}^d \times \Omega \to \mathbb{R}$ that $\forall (t,x)$ satisfies the next equation

Mild solution

$$
\begin{aligned}
u(t,x) = &\int_{\mathbb{R}^d} \mathcal{S}_d(t, x-y) v_0(y)\, dy \\
&+ \frac{\partial}{\partial t}\left(\int_{\mathbb{R}^d} \mathcal{S}_d(t, x-y) u_0(y)\, dy \right) \\
&+ \int_0^t ds \int_{\mathbb{R}^d} \mathcal{S}_d(t-s, x-y) f(s, y, u(s,y))\, dy \\
&+ \int_{(0,t]} d\mu(s) \int_{\mathbb{R}^d} \mathcal{S}_d(t-s, x-y) \sigma(s,y)\, dy\,.
\end{aligned}
$$

Here $\mathcal{S}_d(t,x)$ is the fundamental solution of the wave equation.

We investigate the existence, uniqueness, Hölder continuity and asymptotic behavior of the mild solution.

Mild solution. $d = 1$

Since $\mathcal{S}_1(t,x) = \frac{1}{2a} 1\!\mathrm{I}_{\{|x|<at\}}$, then

$$u(t,x) = \frac{1}{2}(u_0(x+at) - u_0(x-at)) + \frac{1}{2a}\int_{x-at}^{x+at} v_0(y)dy$$
$$+ \frac{1}{2a}\int_0^t ds \int_{x-a(t-s)}^{x+a(t-s)} f(s,y,u(s,y))dy \qquad (3)$$
$$+ \frac{1}{2a}\int_{(0,t]} d\mu(s) \int_{x-a(t-s)}^{x+a(t-s)} \sigma(s,y)dy\,.$$

Assumptions. $d = 1$

A1.1. Functions $u_0(y) = u_0(y, \omega) : \mathbb{R} \times \Omega \to \mathbb{R}$ and $v_0(y) = v_0(y, \omega) : \mathbb{R} \times \Omega \to \mathbb{R}$ are measurable and bounded.

A1.2. $u_0(y)$ is Hölder continuous

$$|u_0(y_1) - u_0(y_2)| \leq C(\omega) |y_1 - y_2|^{\beta(u_0)}, \quad 0 < \beta(u_0) \leq 1 .$$

A1.3. $f(s, y, v) : [0, T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is measurable and bounded.

A1.4. $f(s, y, v)$ is uniformly Lipschitz in $v \in \mathbb{R}$

$$|f(s, y, v_1) - f(s, y, v_2)| \leq C |v_1 - v_2| .$$

Assumptions. $d = 1$

A1.5. $\sigma(s, y) : [0, T] \times \mathbb{R} \to \mathbb{R}$ is measurable and bounded.

A1.6. $\sigma(s, y)$ is Hölder continuous in $s \in [0, T]$, $y \in \mathbb{R}$

$$|\sigma(s_1, y_1) - \sigma(s_2, y_2)| \leq C\Big(|s_1 - s_2|^{\beta(\sigma)} + |y_1 - y_2|^{\beta(\sigma)}\Big),$$
$$1/2 < \beta(\sigma) \leq 1.$$

A1.7. $\forall t \in [0, T] :\quad \Big|\mu\big((0, t]\big)\Big| \leq C(\omega).$

Regularity of stochastic integral in x. $d = 1$

First we consider the stochastic integral from equation (3).

Lemma 1.1 (I. Bodnarchuk, 2017)

Suppose Assumptions A1.5, A1.6 *hold. Then for any fixed* $t \in [0, T]$, $K > 0$, *and* $\tilde{\gamma}_1 < 3/2 - 1/(2\beta(\sigma))$ *the random function*

$$\varphi(x) = \int_{(0,t]} d\mu(s) \int_{x-a(t-s)}^{x+a(t-s)} \sigma(s, y)\, dy, \quad |x| \le K,$$

has a version that is Hölder continuous with exponent $\tilde{\gamma}_1$.

Lemma 1.2 (I. Bodnarchuk, 2017)

Suppose Assumptions $\mathrm{A}1.5 - \mathrm{A}1.7$ *hold. Then for any fixed* $x \in \mathbb{R}$, $\delta > 0$ *and* $\tilde{\gamma}_2 < 1/2$ *the random function*

$$\hat{\varphi}(t) = \int_{(0,t]} d\mu(s) \int_{x-a(t-s)}^{x+a(t-s)} \sigma(s,y)\, dy, \quad t \in [\delta, T],$$

has a version that is Hölder continuous with exponent $\tilde{\gamma}_2$.

The main idea of the proof. $d = 1$

Put for any fixed $t \in [0, T]$:

$$\Delta_{kn}^{(t)} = ((k-1)2^{-n}t, k2^{-n}t], \quad n \geq 0, \quad 1 \leq k \leq 2^n.$$

Consider a function $g(z, s) : Z \times [0, T] \to \mathbb{R}$.
Let $\forall z \in Z$ (Z is an arbitrary set) a function $g(z, \cdot)$ be continuous on $[0, T]$. Put

$$g_n(z, s) = g(z, 0)1\!\!I_{\{0\}}(s) + \sum_{1 \leq k \leq 2^n} g(z, (k-1)2^{-n}T \wedge t)1\!\!I_{\Delta_{kn}^{(T)}}(s).$$

The main idea of the proof. $d = 1$

Then by V. Radchenko, 2015 the function

$$\eta(z) = \int_{(0,t]} g(z,s)d\mu(s)$$

has the version

$$\tilde{\eta}(z) = \int_{(0,t]} g_0(z,s)d\mu(s) + \sum_{n\geq 1}\left(\int_{(0,t]} g_n(z,s)d\mu(s) - \int_{(0,t]} g_{n-1}(z,s)d\mu(s)\right). \quad (4)$$

We use the discrete characterization of Besov spaces that was obtained by A. Kamont, 1997 and get the next estimation for this version (I. Bodnarchuk and G. Shevchenko, 2016)

The main idea of the proof. $d = 1$

$$|\tilde{\eta}(z)| \le |g(z,0)\mu((0,t])| + C\|g(z,\cdot\wedge t)\|_{B^{\alpha}_{22}((0,T])}$$

$$\times \Big\{\sum_{n\ge 1} 2^{-n(2\alpha-1)} \sum_{1\le k\le 2^n} \Big|\mu\big(\Delta^{(T)}_{kn} \cap (0,t]\big)\Big|^2\Big\}^{\frac{1}{2}}, \quad \alpha \in (1/2, 1),$$

where $\|\cdot\|_{B^{\alpha}_{22}((0,t])}$ is the norm of the Besov space on $(0,t]$.

Recall, that the norm of Besov space $B^{\alpha}_{22}([c,d])$, $1/2 < \alpha < 1$ has the form

$$\|g\|_{B^{\alpha}_{22}([c,d])} = \|g\|_{L_2([c,d])} + \left(\int_0^{d-c} (w_2(g,r))^2 r^{-2\alpha-1}\,dr\right)^{1/2},$$

where

$$w_2(g,r) = \sup_{0\le h\le r} \left(\int_c^{d-h} |g(s+h) - g(s)|^2\,ds\right)^{1/2}.$$

The main idea of the proof. $d = 1$

The sum with a stochastic measure is finite by A1.7 and

Lemma 3.1 (V. Radchenko, 2009)

Let $f_l : \mathsf{X} \to \mathbb{R}$, $l \geq 1$, *be measurable functions such that* $\bar{f}(x) = \sum_{l=1}^{\infty} |f_l(x)|$ *is integrable w.r.t.* μ. *Then*

$$\sum_{l=1}^{\infty} \left(\int_{\mathsf{X}} f_l \, d\mu \right)^2 < \infty \quad \text{a. s.}$$

The main result. $d = 1$

Theorem 1.1 (I. Bodnarchuk, 2017)

Suppose Assumptions A1.1 – A1.6 *hold. Then*

1. Equation (2) has a solution $u(t,x)$. *If* $v(t,x)$ *is another solution of (2), then for each* t *and* x: $u(t,x) = v(t,x)$ *a.s.*

2. If in addition Assumption A1.7 *holds than for any fixed* $\delta > 0$, $K > 0$ *and* $\gamma_1, \gamma_2 \in [0, \beta(u_0)]$, $\gamma_1 < 3/2 - 1/(2\beta(\sigma))$, $\gamma_2 < 1/2$ *the stochastic function* $u(t,x)$ *has a vertion* $\tilde{u}(t,x)$ *such that for some* $C(\omega) > 0$

$$|\tilde{u}(t_1,x_1) - \tilde{u}(t_2,x_2)| \leq C(\omega)(|t_1 - t_2|^{\gamma_2} + |x_1 - x_2|^{\gamma_1}),$$
$$t_i \in [\delta, T], \quad x_i \in [-K, K], \quad i = 1, 2.$$

The main idea of the proof. $d = 1$

To prove the Theorem we use the iteration process where $u^{(0)}(t,x) = 0$ and

$$
\begin{aligned}
& u^{(n+1)}(t,x) \\
& \quad = \frac{1}{2}\left(u_0(x+at) - u_0(x-at)\right) + \frac{1}{2a}\int_{x-at}^{x+at} v_0(y)dy \\
& \qquad + \frac{1}{2a}\int_0^t ds \int_{x-a(t-s)}^{x+a(t-s)} f(s,y,u^{(n)}(s,y))dy \\
& \qquad\qquad + \frac{1}{2a}\int_{(0,t]} d\mu(s) \int_{x-a(t-s)}^{x+a(t-s)} \sigma(s,y)dy\,,
\end{aligned}
$$

the method of mathematical induction, and Lemmas 1.1, 1.2. The solution is constructed in the form

$$
u(t,x) = \operatorname{P}\lim_{n\to\infty} u^{(n)}(t,x), \quad \forall (t,x) \in [0,T]\times\mathbb{R}.
$$

In addition to (2), consider the following problems:

$$\begin{cases} \dfrac{\partial^2 u_j(t,x)}{\partial t^2} = a^2 \Delta_x u_j(t,x) + f_j(t,x,u_j(t,x)) + \sigma_j(t,x)\dot{\mu}(t), \\ u_j(0,x) = u_{0j}(x); \quad \dfrac{\partial u_j(0,x)}{\partial t} = v_{0j}(x), \end{cases}$$

where $j \geq 1$. The solutions of these problems are considered in the mild sense, that is,

$$\begin{aligned} u_j(t,x) = \frac{1}{2}(u_{0j}(x+at) - u_{0j}(x-at)) + \frac{1}{2a}\int_{x-at}^{x+at} v_{0j}(y)dy \\ + \frac{1}{2a}\int_0^t ds \int_{x-a(t-s)}^{x+a(t-s)} f_j(s,y,u_j(s,y))dy \\ + \frac{1}{2a}\int_{(0,t]} d\mu(s) \int_{x-a(t-s)}^{x+a(t-s)} \sigma_j(s,y)dy\,. \end{aligned} \tag{5}$$

Assumptions*. $d = 1$

A1.1*. Functions $u_{0j}(y) = u_{0j}(y, \omega) : \mathbb{R} \times \Omega \to \mathbb{R}$ and $v_{0j}(y) = v_{0j}(y, \omega) : \mathbb{R} \times \Omega \to \mathbb{R}$ are measurable and bounded.

A1.2*. $u_{0j}(y)$ is Hölder continuous

$$|u_{0j}(y_1) - u_{0j}(y_2)| \leq C(\omega) \, |y_1 - y_2|^{\beta(u_0)} , \quad 0 < \beta(u_0) \leq 1 .$$

A1.3*. $f_j(s, y, v) : [0, T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is measurable and bounded.

A1.4*. $f_j(s, y, v)$ is uniformly Lipschitz in $v \in \mathbb{R}$

$$|f_j(s, y, v_1) - f_j(s, y, v_2)| \leq C \, |v_1 - v_2| \ .$$

Assumptions*. $d = 1$

A1.5*. $\sigma_j(s, y) : [0, T] \times \mathbb{R} \to \mathbb{R}$ is measurable and bounded.

A1.6*. $\sigma_j(s, y)$ is Hölder continuous in $s \in [0, T]$, $y \in \mathbb{R}$

$$|\sigma_j(s_1, y_1) - \sigma_j(s_2, y_2)| \le C\Big(|s_1 - s_2|^{\beta(\sigma)} + |y_1 - y_2|^{\beta(\sigma)}\Big),$$
$$1/2 < \beta(\sigma) \le 1.$$

A1.7. $\forall t \in [0, T]$: $\quad \Big|\mu\big((0, t]\big)\Big| \le C(\omega)$.

Theorem 1.2 (I. Bodnarchuk, 2017)

Let the components of equations (3) and (5) satisfy Assumptions A1.1 – A1.7 *and* A1.1* – A1.6*, A1.7 *respectively, for all* $j \geq 1$. *Also let*

$$U_j = \sup_{y \in \mathbb{R}} |u_{0j}(y) - u_0(y)| \to 0 \quad \text{a.s.},$$

$$V_j = \sup_{y \in \mathbb{R}} |v_{0j}(y) - v_0(y)| \to 0 \quad \text{a.s.},$$

$$\Sigma_j = \sup_{(s,y) \in [0,T] \times \mathbb{R}} |\sigma_j(s,y) - \sigma(s,y)| \to 0,$$

$$F_j = \sup_{(s,y,v) \in [0,T] \times \mathbb{R} \times \mathbb{R}} |f_j(s,y,v) - f(s,y,v)| \to 0, \quad j \to \infty.$$

Then for all $\delta > 0$, $(t,x) \in [\delta, T] \times \mathbb{R}$:

$$|u_j(t,x) - u(t,x)| \to 0, \quad j \to \infty \quad \text{a.s.}$$

Asymptotic behavior. $d = 1$

A1.8. $|u_0(y)| \to 0$ a.s., $|v_0(y)| \to 0$ a.s.,
$$\sup_{s\in[0,T],v\in\mathbb{R}} |f(s,y,v)| \to 0, \ \sup_{s\in[0,T]} |\sigma(s,y)| \to 0, \ |y| \to \infty.$$

Theorem 1.3 (I. Bodnarchuk, 2018)

Suppose Assumptions A1.1 – A1.6 *and A1.8 hold. Than the stochastic function* $u(t,x)$ *has a vertion* $\tilde{u}(t,x)$ *such that for all* $t \in [0,T]$, $\omega \in \Omega$

$$|u(t,x)| \to 0, \quad |x| \to \infty.$$

Mild solution. $d = 2$

Since $\mathcal{S}_2(t,x) = \frac{1}{2a\pi}\left(a^2t^2 - |x|^2\right)^{-1/2} 1\!\!1_{\{|x|<at\}}$, then

$$
\begin{aligned}
u(t,x) = &\frac{1}{2a\pi}\int_{B(x,at)} \frac{v_0(y)}{\sqrt{a^2t^2 - |x-y|^2}}\,dy + \\
&+ \frac{\partial}{\partial t}\left(\frac{1}{2a\pi}\int_{B(x,at)} \frac{u_0(y)}{\sqrt{a^2t^2 - |x-y|^2}}\,dy\right) + \\
&+ \frac{1}{2a\pi}\int_0^t ds \int_{B(x,a(t-s))} \frac{f(s,y,u(s,y))}{\sqrt{a^2(t-s)^2 - |x-y|^2}}\,dy + \\
&+ \frac{1}{2a\pi}\int_{(0,t]} d\mu(s) \int_{B(x,a(t-s))} \frac{\sigma(s,y)}{\sqrt{a^2(t-s)^2 - |x-y|^2}}\,dy\,,
\end{aligned}
\qquad (6)
$$

where $B(x,r) = \{y \in \mathbb{R}^2 : |x-y| < r\}$.

Assumptions. $d=2$

A2.1. Functions $u_0(y)=u_0(y,\omega):\mathbb{R}^2\times\Omega\to\mathbb{R}$ and $v_0(y)=v_0(y,\omega):\mathbb{R}^2\times\Omega\to\mathbb{R}$ are measurable and bounded.

A2.2. $v_0(y), u_0(y), \frac{\partial u_0(y)}{\partial y_i}, i=1,2$ are Hölder continuous

$$|v_0(y')-v_0(y'')|\le L_{v_0}(\omega)\,|y'-y''|^{\beta(v_0)},\ 0<\beta(v_0)\le 1;$$

$$|u_0(y')-u_0(y'')|\le L_{u_0}(\omega)\,|y'-y''|^{\beta(u_0)},\ 0<\beta(u_0)\le 1;$$

$$\left|\frac{\partial u_0}{\partial y_i}(y')-\frac{\partial u_0}{\partial y_i}(y'')\right|\le L_{u_0}(\omega)\,|y'-y''|^{\beta(u_0)}\ .$$

A2.3. $f(s,y,v):[0,T]\times\mathbb{R}^2\times\mathbb{R}\to\mathbb{R}$ is measurable and bounded.

A2.4. $f(s,y,v)$ is uniformly Lipschitz in $y\in\mathbb{R}^2, v\in\mathbb{R}$

$$|f(s,y_1,v_1)-f(s,y_2,v_2)|\le C\left(|y_1-y_2|+|v_1-v_2|\right)\ .$$

Assumptions. $d = 2$

A2.5. $\sigma(s, y) : [0, T] \times \mathbb{R}^2 \to \mathbb{R}$ is measurable and bounded.

A2.6. $\sigma(s, y)$ is Hölder continuous in $s \in [0, T]$, $y \in \mathbb{R}^2$

$$|\sigma(s_1, y_1) - \sigma(s_2, y_2)| \le C\Big(|s_1 - s_2|^{\beta(\sigma)} + |y_1 - y_2|^{\beta(\sigma)}\Big),$$
$$1/2 < \beta(\sigma) \le 1.$$

A2.7. $\forall t \in [0, T]: \quad \Big|\mu\big((0, t]\big)\Big| \le C(\omega)$.

The main result. $d = 2$

Theorem 2.1 (I. Bodnarchuk, B. Radchenko, 2018)

Suppose Assumptions A2.1 – A2.6 *hold. Then*

1. Equation (6) has a solution $u(t,x)$. *If* $v(t,x)$ *is another solution of (6), then for each* t *and* x: $u(t,x) = v(t,x)$ *a.s.*

2. If in addition Assumption A2.7 *holds than for any fixed* $\delta > 0$, $K > 0$ *and* $\gamma \in [0, \beta(v_0) \wedge \beta(u_0)]$, $\gamma < \beta(\sigma) - 1/2$ *the stochastic function* $u(t,x)$ *has a vertion* $\tilde{u}(t,x)$ *such that for some* $C(\omega) > 0$

$$|\tilde{u}(t_1, x') - \tilde{u}(t_2, x'')| \leq C(\omega)(|t_1 - t_2|^{\gamma} + |x' - x''|^{\gamma}),$$
$$t_i \in [\delta, T], \quad x', x'' \in \bar{B}(0, K), \quad i = 1, 2.$$

Continuous dependence on data. $d = 2$

In addition to (2), consider the following problems:

$$\begin{cases} \dfrac{\partial^2 u_j(t,x)}{\partial t^2} = a^2 \Delta_x u_j(t,x) + f_j(t,x,u_j(t,x)) + \sigma_j(t,x)\dot{\mu}(t), \\ u_j(0,x) = u_{0j}(x); \quad \dfrac{\partial u_j(0,x)}{\partial t} = v_{0j}(x), \end{cases}$$

where $j \geq 1$. The solutions of these problems are considered in the mild sense, that is,

$$\begin{aligned} u_j(t,x) = \frac{1}{2}(u_{0j}(x+at) - u_{0j}(x-at)) + \frac{1}{2a}\int_{x-at}^{x+at} v_{0j}(y)dy \\ + \frac{1}{2a}\int_0^t ds \int_{x-a(t-s)}^{x+a(t-s)} f_j(s,y,u_j(s,y))dy \qquad (7) \\ + \frac{1}{2a}\int_{(0,t]} d\mu(s) \int_{x-a(t-s)}^{x+a(t-s)} \sigma_j(s,y)dy\,. \end{aligned}$$

Assumptions*. $d = 2$

A2.1*. Functions $u_{0j}(y) = u_{0j}(y,\omega) : \mathbb{R}^2 \times \Omega \to \mathbb{R}$ and $v_{0j}(y) = v_{0j}(y,\omega) : \mathbb{R}^2 \times \Omega \to \mathbb{R}$ are measurable and bounded.

A2.2*. $v_{0j}(y), u_{0j}(y), \frac{\partial u_{0j}(y)}{\partial y_i}, i = 1, 2$ are Hölder continuous

$$|v_0(y') - v_0(y'')| \leq L_{v_0}(\omega)\,|y' - y''|^{\beta(v_0)}\,,\ 0 < \beta(v_0) \leq 1;$$

$$|u_0(y') - u_0(y'')| \leq L_{u_0}(\omega)\,|y' - y''|^{\beta(u_0)}\,,\ 0 < \beta(u_0) \leq 1\,;$$

$$\left|\frac{\partial u_0}{\partial y_i}(y') - \frac{\partial u_0}{\partial y_i}(y'')\right| \leq L_{u_0}(\omega)\,|y' - y''|^{\beta(u_0)}\ .$$

A2.3*. $f_j(s,y,v) : [0,T] \times \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}$ is measurable and bounded.

A2.4*. $f_j(s,y,v)$ is uniformly Lipschitz in $y \in \mathbb{R}^2$, $v \in \mathbb{R}$

$$|f_j(s,y_1,v_1) - f_j(s,y_2,v_2)| \leq L_f\left(|y_1 - y_2| + |v_1 - v_2|\right)\ .$$

Assumptions*. $d = 2$

A2.5*. $\sigma_j(s, y) : [0, T] \times \mathbb{R}^2 \to \mathbb{R}$ is measurable and bounded.

A2.6*. $\sigma_j(s, y)$ is Hölder continuous in $s \in [0, T]$, $y \in \mathbb{R}^2$

$$|\sigma_j(s_1, y_1) - \sigma_j(s_2, y_2)| \le C\Big(|s_1 - s_2|^{\beta(\sigma)} + |y_1 - y_2|^{\beta(\sigma)}\Big),$$
$$1/2 < \beta(\sigma) \le 1.$$

A2.7. $\forall t \in [0, T]: \quad \Big|\mu\big((0, t]\big)\Big| \le C(\omega).$

Theorem 2.2 (I. Bodnarchuk, B. Radchenko, 2018)

Let the components of equations (6) and (7) satisfy Assumptions A2.1 – A2.7 *and* A2.1* – A2.6*, A2.7 *respectively, for all* $j \geq 1$. *Also let*

$$V_j = \sup_{y \in \mathbb{R}^2} |v_{0j}(y) - v_0(y)| \to 0 \quad a.s.,$$

$$U_j = \sup_{y \in \mathbb{R}^2} |u_{0j}(y) - u_0(y)| \to 0 \quad a.s.,$$

$$Du_j = \sup_{y \in \mathbb{R}^2, i=1,2} \left| \frac{\partial u_{0j}}{\partial y_i}(y) - \frac{\partial u_0}{\partial y_i}(y) \right| \to 0 \quad a.s.$$

$$\Sigma_j = \sup_{(s,y) \in [0,T] \times \mathbb{R}^2} |\sigma_j(s, y) - \sigma(s, y)| \to 0,$$

$$F_j = \sup_{(s,y,v) \in [0,T] \times \mathbb{R}^2 \times \mathbb{R}} |f_j(s, y, v) - f(s, y, v)| \to 0, j \to \infty.$$

Then for all $\delta > 0$, $(t, x) \in [\delta, T] \times \mathbb{R}^2$:

$$|u_j(t, x) - u(t, x)| \to 0, \quad j \to \infty \quad a.s.$$

Asymptotic behavior. $d = 2$

A2.8. $|u_0(y)| \to 0$ a.s., $\left|\frac{\partial u_0}{\partial y_i}(y)\right| \to 0$ a.s., $|v_0(y)| \to 0$ a.s.,
$$\sup_{s\in[0,T],v\in\mathbb{R}^2} |f(s,y,v)| \to 0, \ \sup_{s\in[0,T]} |\sigma(s,y)| \to 0, \ |y| \to \infty.$$

Theorem 2.3 (I. Bodnarchuk, 2018)

Suppose Assumptions A2.1 – A2.6 *and A2.8 hold. Than the stochastic function* $u(t,x)$ *has a vertion* $\tilde{u}(t,x)$ *such that for all* $t \in [0,T]$, $\omega \in \Omega$

$$|u(t,x)| \to 0, \quad |x| \to \infty.$$

Mild solution. $d = 3$

Since $\mathcal{S}_3(t,x) = \dfrac{1}{4a^2\pi t}\delta_{S_{at}} 1\!\mathrm{I}_{\{t>0\}}$, then

$$
\begin{aligned}
u(t,x) &= \frac{1}{4a^2\pi t} \int\limits_{|y-x|=at} v_0(y)\, dS(y) \\
&+ \frac{1}{4a^2\pi} \frac{\partial}{\partial t} \left(\frac{1}{t} \int\limits_{|y-x|=at} u_0(y)\, dS(y) \right) \\
&+ \frac{1}{4a^2\pi} \int_0^t ds \int\limits_{|y-x|=a(t-s)} \frac{f(s,y,u(s,y))}{t-s}\, dS(y) \\
&+ \frac{1}{4a^2\pi} \int_0^t \frac{1}{t-s} d\mu(s) \int_{|x-y|=a(t-s)} \sigma(s,y)\, dS(y)\,.
\end{aligned}
\tag{8}
$$

Assumptions. $d = 3$

A3.1. Functions $u_0(y) = u_0(y, \omega) : \mathbb{R}^3 \times \Omega \to \mathbb{R}$ and $v_0(y) = v_0(y, \omega) : \mathbb{R}^3 \times \Omega \to \mathbb{R}$ are measurable and bounded.

A3.2. $v_0(y), u_0(y), \frac{\partial u_0(y)}{\partial y_i}, i = 1, 2, 3$ are Hölder continuous

$$|v_0(y') - v_0(y'')| \leq L_{v_0}(\omega) \, |y' - y''|^{\beta(v_0)}, \; 0 < \beta(v_0) \leq 1;$$

$$|u_0(y') - u_0(y'')| \leq L_{u_0}(\omega) \, |y' - y''|^{\beta(u_0)}, \; 0 < \beta(u_0) \leq 1;$$

$$\left| \frac{\partial u_0}{\partial y_i}(y') - \frac{\partial u_0}{\partial y_i}(y'') \right| \leq L_{u_0}(\omega) \, |y' - y''|^{\beta(u_0)} \; .$$

A3.3. $f(s, y, v) : [0, T] \times \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}$ is measurable and bounded.

A3.4. $f(s, y, v)$ is uniformly Lipschitz in $y \in \mathbb{R}^3, v \in \mathbb{R}$

$$|f(s, y_1, v_1) - f(s, y_2, v_2)| \leq C \left(|y_1 - y_2| + |v_1 - v_2| \right) \; .$$

Assumptions. $d = 3$

A3.5. $\sigma(s,y) : [0,T] \times \mathbb{R}^3 \to \mathbb{R}$ is measurable and bounded.

A3.6. $\sigma(s,y)$ is Hölder continuous in $s \in [0,T]$, $y \in \mathbb{R}^3$

$$|\sigma(s_1,y_1) - \sigma(s_2,y_2)| \le C\big(|s_1 - s_2|^{\beta(\sigma)} + |y_1 - y_2|^{\beta(\sigma)}\big),$$
$$1/2 < \beta(\sigma) \le 1.$$

A3.7. $\forall t \in [0,T]: \quad \big|\mu\big((0,t]\big)\big| \le C(\omega)$.

The main result. $d = 3$

Theorem 3.1

Suppose Assumptions A3.1 – A3.6 *hold. Then*

1. Equation (8) has a solution $u(t, x)$. *If* $v(t, x)$ *is another solution of (8), then for each* t *and* x: $u(t, x) = v(t, x)$ *a.s.*

2. If in addition Assumption A3.7 *holds than for any fixed* $\delta > 0,\ K > 0$ *and* $\gamma \in [0, \beta(v_0) \wedge \beta(u_0)]$, $\gamma < \beta(\sigma) - 1/2$ *the stochastic function* $u(t, x)$ *has a vertion* $\tilde{u}(t, x)$ *such that for some* $C(\omega) > 0$

$$|\bar{u}(t_1, x') - \bar{u}(t_2, x'')| \leq C(\omega) \left(|t_1 - t_2|^\gamma + |x' - x''|^\gamma\right), \\ t_1, t_2 \in [\delta, T], x', x'' \in \bar{B}(0, K).$$

References

- R.C. Dalang, The stochastic wave equation, A minicourse on stochastic partial differential equations. Lecture Notes in Math. **1962** (2009), 39–71.
- J.L. Davis, *Wave Propagation in Solids and Fluids*, Springer, Verlag, 1988.
- S. Kwapień, W.A. Woyczyński, *Random Series and Stochastic Integrals: Single and Multiple*, Birkhäuser, Boston, 1992.
- L. Drewnowski, Topological rings of sets, continuous set functions, integration. III, Bull. Acad. Pol. Sci. Sér. Sci. Math. Astron. Phys. **20** (1972), 439–445.
- T. Memin, Y. Mishura, and E. Valkeila, Inequalities for the moments of Wiener integrals with respect to a fractional Brownian motion, Statist. Probab. Lett. **51** (2001), 197–206.
- I.M. Bodnarchuk, *Wave equation with a stochastic measure*, Theory Probab. Math. Statist. **94** (2017), 1–16.

References

◦ V.M. Radchenko, *Evolution equations driven by general stochastic measures in Hilbert space*, Theory Probab. Appl. **59** (2015), №2, 328 – 339.

◦ A. Kamont, *A discrete characterization of Besov spaces*, Approx. Theory Appl. **13** (1997), №2, 63–77.

◦ I.M. Bodnarchuk, G.M. Shevchenko, *Heat equation in a multidimensional domain with a general stochastic measure*, Theory Probab. Math. Statist. **93** (2016), 19–31.

◦ I.M. Bodnarchuk, *Asymptotics of the mild solution for the wave equation with a stochastic measure*, Vesnik of Brest University. Series 4. Physics. Mathematics. **2** (2018), 70–78. (Russian)

◦ I.M. Bodnarchuk, V.M. Radchenko, *Wave equation in a plane driven by a general stochastic measure*, Teor. Imov. Matem. Statyst. **98** (2018), 70–86. (Ukrainian)

Thank you!