Quasi-stationarity with moving boundaries

Stochastic Processes and Statistical Machine Learning I

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February 15, 2018

I. Framework and definitions : when boundaries are NOT moving

Framework : Absorbed Markov processes

$(X_t)_{t\geq 0}$ Markov process evolving in $E \cup A$, where A is considered as a trap for $(X_t)_{t\geq 0}$:

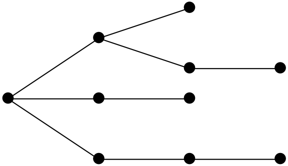
$$X_t \in A, \qquad \forall t > \tau_A$$

where

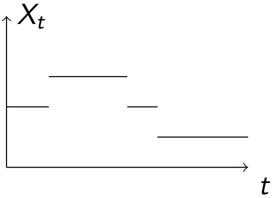
$$\tau_A := \inf\{t \geq 0 : X_t \in A\}$$

Examples

Galton-Watson process



Birth-death process



$X_0 = 1 \; X_1 = 3 \; X_2 = 4 \; X_3 = 2$

.

More generally : any Markov processes stopped when reaching a given subset of the state space

Asymptotic behavior

- If $\tau_A < \infty$ $\mathbb{P}_x$-almost surely for any $x \in E$, $(X_t)_{t\geq 0}$ will live in A as t goes to infinity
- When τ_A is exceptionally big, a meta-stable state can appear before the Markov process is absorbed
- To characterize this meta-stable state, the idea is to study the asymptotic behavior of

$$\mathbb{P}_x(X_t \in \cdot | \tau_A > t) \tag{1}$$

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Questions

- Is there weak convergence of (1)? For which $x \in E$?
- What is the limit?

Quasi-limit distribution

Main assumptions :

- $\mathbb{P}_x(\tau_A < \infty) = 1, \quad \forall x \in E$
- $\mathbb{P}_x(\tau_A > t) > 0, \quad \forall x \in E, \forall t \geq 0$

Definition : Quasi-limit distribution (QLD)

α is a *quasi-limit distribution (QLD)* if, for some initial distribution μ,

$$\alpha = \lim_{t \to \infty} \mathbb{P}_\mu(X_t \in \cdot | \tau_A > t)$$

Digression - Stationary distribution: If, for some initial law μ, $\mathbb{P}_\mu(X_t \in \cdot)$ converges weakly, then π defined by

$$\pi := \lim_{t \to \infty} \mathbb{P}_\mu(X_t \in \cdot)$$

is a stationary distribution of $(X_t)_{t \geq 0}$, i.e. a prob. measure satisfying

$$\mathbb{P}_\pi(X_t \in \cdot) = \pi, \quad \forall t \geq 0$$

Quasi-stationary distribution

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Definition : Quasi-stationary distribution (QSD)

α is a *quasi-stationary distribution (QSD)* if

$$\mathbb{P}_\alpha(X_t \in \cdot | \tau_A > t) = \alpha, \qquad \forall t \geq 0$$

Some remarks

Equivalence between QSD and QLD

$$\text{QLD} \Longleftrightarrow \text{QSD}$$

- $\Longleftarrow$: Obvious since, for $\mu = \alpha$, $\mathbb{P}_\mu(X_t \in \cdot | \tau_A > t) = \alpha$
- $\Longrightarrow$: Denote by

$$\mu_t = \mathbb{P}_\mu(X_t \in \cdot | \tau_A > t)$$

According to Markov property,

$$\mu_{t+s} = \mathbb{P}_{\mu_s}(X_t \in \cdot | \tau_A > t)$$

Argument of fixed point theorem : $\alpha = \lim_{s \to \infty} \mu_s$ satisfies

$$\alpha = \mathbb{P}_\alpha(X_t \in \cdot | \tau_A > t), \qquad \forall t \geq 0$$

Definition : Q-process

Q-process

We say that $(Y_t)_{t\geq 0}$ is a Q-process if, for any initial law μ,

$$\mathbb{P}_\mu(Y_{[0,s]} \in \cdot) = \lim_{t\to\infty} \mathbb{P}_\mu(X_{[0,s]} \in \cdot | \tau_A > t), \quad \forall s \geq 0$$

- The Q-process can be considered as the law of the process X conditioned never to be absorbed by A.
- The Q-process is a Markov process.
- For some processes, Q-process exists without having existence of QSD (ex : Brownian motion stopped at 0)

Quasi-ergodic distribution

Mean ergodic theorem for Markov processes: If π is a stationary measure, then under some assumptions on X for any measurable function f,

$$\frac{1}{t}\int_0^t f(X_s)ds \underset{t\to\infty}{\longrightarrow} \int fd\pi, \qquad \text{almost surely}$$

Definition : Quasi-ergodic distribution (QED)

β is a *quasi-ergodic distribution (QED)* if for some prob. meas. μ,

$$\beta = \lim_{t\to\infty}\frac{1}{t}\int_0^t \mathbb{P}_\mu(X_s \in \cdot | \tau_A > t)ds$$

Remark: The QED is different from the QSD

Spectral property of QSD

- Sub-Markovian semi-group of $(X_t)_{t\geq 0}$:

$$P_t f(x) = \mathbb{E}_x(f(X_t)\mathbb{1}_{\tau_A > t})$$

- α QSD $\Leftrightarrow \int_E P_t f(x)\alpha(dx) = e^{-\lambda t}\int_E f(x)\alpha(dx) \quad (\lambda > 0)$
- **Comparison with stationary distribution :**

$$\pi \text{ stationary distribution} \Leftrightarrow \int \mathbb{E}_x(f(X_t))\pi(dx) = \int f(x)\pi(dx)$$

II. Quasi-stationarity with moving boundaries

- Sometimes $(X_t)_{t\geq 0}$ can be absorbed by a moving trap
- Example : Cattiaux-Christophe-Gadat, 2016

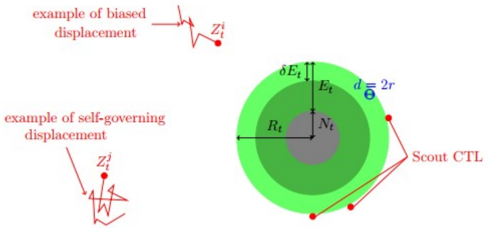

Figure: Figure 1 from "A stochastic model for cytotoxic T. lymphocyte interaction with tumor nodules"

Questions

$(X_t)_{t\geq 0}$ evolving in $E_t \cup A_t$ and absorbed in $(A_t)_{t\geq 0}$.

$$\tau_A = \inf\{t \geq 0 : X_t \in A_t\}$$

$$\tau_{A\circ\theta_s} = \inf\{t \geq 0 : X_t \in A_{t+s}\}$$

Questions

Can we still define the notion of

- QSD ?
- QLD ?
- Q-process ?
- QED ?

For which behavior of $(A_t)_{t\geq 0}$?

(Irr) : $\forall t \geq 0, \forall x, y \in E_t, \forall \epsilon > 0, \exists u \geq 0, \mathbb{P}_x(X_{u \wedge \tau_{A_t}} \in B(y, \epsilon)) > 0$

where $\tau_{A_t} = \inf\{u \geq 0 : X_u \in A_t\}$ and $B(y, \epsilon) =$ ball of center y and radius ϵ

Proposition (O., 2017)

Under the assumption of irreducibility **(Irr)**, for any $s \geq 0$, there is no prob. measure α s.t.

$$\alpha = \mathbb{P}_\alpha(X_t \in \cdot | \tau_{A \circ \theta_s} > t)$$

Proof of the proposition

Proof in discrete-time setting: For any μ and $n \geq 0$, denote by

$$\mu_n := \mathbb{P}_\mu(X_n \in \cdot | \tau_A > n)$$

Then, according to Markov property, for any $n \geq 1$,

$$\mu_n = \mathbb{P}_{\mu_{n-1}}(X_1 \in \cdot | \tau_{A_n} > 1)$$

where $\tau_{A_n} = \inf\{m \geq 0 : X_m \in A_n\}$. Thus, if $\mu_0 = \alpha$ satisfies Prop 1, then $\mu_n = \alpha$ for all n and

$$\alpha = \mathbb{P}_\alpha(X_1 \in \cdot | \tau_{A_n} > 1), \qquad \forall n \geq 1$$

which will imply that *Supp* $\alpha = E_n$ for any n : Impossible !

QSD and QLD aren't equivalent anymore

- Even if we cannot define QSD when the absorbing set moves, QLD can still exist in certain case.
- Example : Assume that $A_n = A_{n_0}$ for any $n \geq n_0$. Then, By Markov property,

$$\mathbb{P}_\mu(X_{n+n_0} \in \cdot | \tau_A > n + n_0) = \mathbb{P}_{\phi_{n_0}(\mu)}(X_n \in \cdot | \tau_{A_{n_0}} > n)$$

where

$$\phi_{n_0} : \mu \to \mathbb{P}_\mu(X_{n_0} \in \cdot | \tau_A > n_0)$$

Hence $\mathbb{P}_\mu(X_n \in \cdot | \tau_A > n)$ converges if

$$\mu \in \{\nu \text{ prob. meas.} : \mathbb{P}_{\phi_{n_0}(\nu)}(X_n \in \cdot | \tau_{A_{n_0}} > n)\}$$

- Q-process and quasi-ergodic distribution can also still make sense with moving boundaries

III. Q-process and quasi-ergodic distribution : Champagnat-Villemonais condition

Time-inhomogeneous semi-group

- Sub-Markovian time-inhomogeneous semi-group of $(X_t)_{t\geq 0}$:

$$P_{s,t}f(x) = \mathbb{E}_x(f(X_{t-s})\mathbb{1}_{\tau_{A\circ\theta_s}>t-s})$$

- It is very difficult to use spectral techniques to characterize QLD? QED and Q-process

Champagnat-Villemonais condition

Consider A as a non-moving boundaries

Champagnat-Villemonais condition (CV)

CV1 there exists $\nu \in \mathcal{M}_1(E)$, $t_0, c_1 > 0$ s.t.

$$\mathbb{P}_x(X_{t_0} \in \cdot | \tau_A > t_0) \geq c_1 \nu, \quad \forall x \in E$$

CV2 there exists $c_2 > 0$ s.t.

$$\mathbb{P}_\nu(\tau_A > t) \geq c_2 \mathbb{P}_x(\tau_A > t), \quad \forall x \in E, \forall t \geq 0$$

Exponential quasi-ergodicity (non-moving case)

Theorem (Champagnat-Villemonais, 2016)

(CV1) and (CV2) $\Leftrightarrow$ *there exist* $C, \gamma > 0$ *s.t. for any initial law* μ *and any* $t \geq 0$,

$$||\mathbb{P}_\mu(X_t \in \cdot | \tau_A > t) - \alpha||_{TV} \leq Ce^{-\gamma t}$$

where

$$||\mu||_{TV} = \sup_{||f||_\infty \leq 1} \left| \int_E f(x)\mu(dx) \right|$$

(CV1) and (CV2) imply also

- Existence of Q-process
- Existence and uniqueness of QED

Assumption (A)

There exists $(\nu_s)_{s\geq 0}$ prob. measures and t_0, c_1 and $c_2 > 0$ s.t.

A1 For any $s \geq 0$ and $x \in E_s$

$$\mathbb{P}_x(X_{t_0} \in \cdot | \tau_{A\circ\theta_s} > t_0) \geq c_1 \nu_{s+t_0}$$

A2 For any $s, t \geq 0$ and $x \in E_s$,

$$\mathbb{P}_{\nu_s}(\tau_{A\circ\theta_s} > t) \geq c_2 \mathbb{P}_x(\tau_{A\circ\theta_s} > t)$$

Existence of Q-process and exponential decay

Theorem (Champagnat-Villemonais, 2016)

Under Assumption (A), there exists a time-inhomogeneous Markov process $(Y_t)_{t\geq 0}$ s.t. for any $0 \leq s \leq t, \forall x \in E_s$,

$$\mathbb{P}_{s,x}(Y_{[s,s+t]} \in \cdot) = \lim_{T\to\infty} \mathbb{P}_x(X_{[0,t]} \in \cdot | \tau_{A\circ\theta_s} > t + T),$$

Theorem (Champagnat-Villemonais, 2016)

Under Assumption (A), there exists a time-inhomogeneous Markov process $(Y_t)_{t\geq 0}$ s.t. for any $0 \leq s \leq t, \forall x \in E_s$,

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Theorem (O., 2018)

For any $s, t \geq 0$ and $x \in E_s$, there exist $d' \in (0,1)$ and $C_{s,t,x} > 0$ such that for any $T \geq 0$,

$$||\mathbb{P}_x(X_{[0,t]} \in \cdot | \tau_{A\circ\theta_s} > t + T) - \mathbb{P}_{s,x}(Y_{[s,s+t]} \in \cdot)||_{TV} \leq C_{s,t,x}(1-d')^{\left\lfloor \frac{T}{t_{max}} \right\rfloor}$$

Quasi-ergodic distribution

Corollary (O.,2018)

Furthermore, if

1. $\forall x, s, \quad \sup_{t\geq 0} C_{s,t,x} < \infty$
2. $\forall \mu, \quad \frac{1}{t}\int_0^t \mathbb{P}_\mu(Y_s \in \cdot)ds \underset{t\to\infty}{\longrightarrow} \beta$

Then for any initial law μ

$$\frac{1}{t}\int_0^t \mathbb{P}_\mu(X_s \in \cdot | \tau_A > t)ds \underset{t\to\infty}{\longrightarrow} \beta$$

Proof.

$$\left|\left|\frac{1}{t}\int_0^t \mathbb{P}_\mu(X_s \in \cdot|\tau_A > t) - \beta\right|\right|_{TV}$$
$$\leq \frac{1}{t}\int_0^t ||\mathbb{P}_\mu(X_s \in \cdot|\tau_A > t) - \mathbb{P}_\mu(Y_s \in \cdot)||_{TV}\, ds$$
$$+ \left|\left|\frac{1}{t}\int_0^t \mathbb{P}_\mu(Y_s \in \cdot)ds - \beta\right|\right|_{TV}$$
$$\leq \frac{C}{t} + \left|\left|\frac{1}{t}\int_0^t \mathbb{P}_\mu(Y_s \in \cdot)ds - \beta\right|\right|_{TV} \xrightarrow[t\to\infty]{} 0$$

Two types of behavior

- A is γ-periodic

Theorem (O.,2018)

If Assumption (A) holds and $t_0 \in \gamma\mathbb{N}$, then there exists β such that for any initial law μ

$$\frac{1}{t}\int_0^t \mathbb{P}_\mu(X_s \in \cdot | \tau_A > t) ds \underset{t\to\infty}{\longrightarrow} \beta$$

- A is a non-increasing nested sequence (i.e. $A_t \subset A_s, \forall s \leq t$) converging towards A_∞.

Theorem (O.,2018)

If Assumption (A) holds and (CV) holds for A_∞, then there exists β such that for any initial law μ

$$\frac{1}{t}\int_0^t \mathbb{P}_\mu(X_s \in \cdot | \tau_A > t) ds \underset{t\to\infty}{\longrightarrow} \beta$$

A few words about QLD

- A γ-periodic

Proposition (O.,2017)

If (Irr) holds, then for any initial law μ, the sequence

$$\mathbb{P}_\mu(X_t \in \cdot | \tau_A > t)$$

does not converge.

- A is a non-increasing nested sequence (i.e. $A_t \subset A_s, \forall s \leq t$) converging towards A_∞.

Theorem (O.,2018)

If Assumption (A) holds and (CV) holds for A_∞, then there exists α such that for any initial law μ

$$\mathbb{P}_\mu(X_t \in \cdot | \tau_A > t) \underset{t \to \infty}{\longrightarrow} \alpha$$

I'm done ! Thank you for your attention !