Zeta functions everywhere !

It has always been a motivating problem for mathematicians to understand why two questions, emerging from quite different setups, appear to be deeply connected. Here are two instances of such related questions.

Zeta functions and Galois module structures.

Philippe Cassou-Noguès
Mathematics Institute, Bordeaux

In this talk I shall address a question raised by A. Fröhlich in the seventies, putting it in a historical context. The underlying problem being of arithmetic nature, I shall first recall some basic concepts such as that of a Galois extension. Here is the question: starting from a Galois extension $N/\mathbf{Q}$, does the ring of integers $\mathcal{O}_N$ in N have a $\mathbf{Z}$-basis of the form $\{a^\gamma\}$, γ running over $\Gamma = \mathrm{Gal}(N/\mathbf{Q})$?
This question was answered positively by Hilbert when Γ is commutative. As I shall explain in this talk, the obstruction to the existence of such a basis, when Γ is non commutative, comes–as conjectured by Fröhlich– from the existence of zeroes at $s = 1/2$ of some analytic functions attached to the extension. If time permits, I shall mention some of the techniques developped to solve this problem, since they have since then led to new research in Number Theory and Arithmetic Geometry.

Zeta functions and roots of Bernstein polynomials.

Pierrette Cassou-Noguès
Mathematics Institute, Bordeaux

This talk which takes place in the p-adic setup– and for which I will provide the basic concepts needed here– addresses rationality issues.
In 1966, Borewicz and Safarevic conjectured that the generating series $P(t)$ for the number of solutions of a certain polynomial equation over a p-adic field K, is a rational fonction of t. Igusa proved this rationality property in the 70's using the Igusa Zeta function $Z_K(s)$ and the equality $P(t) = \frac{1-t\,Z_K(s)}{1-t}$ where $t = q^{-s}$. When $K = \mathbb{R}$ or $\mathbb{C}$, Igusa conjectured that the poles of $Z_K(s)$ are roots of the Bernstein polynomial of f.

This conjecture was proven for $n = 2$ by F. Loeser in 1988. We shall discuss it in the frame of the motivic Zeta function (defined by Denef and Loeser) which is easier to handle than the Igusa Zeta function. For this purpose, we shall use a very old tool, the Newton polygon, in order to give an idea of the ideas underlying the proof.