Non-linear SPDEs, modelled distributions and renormalisation
Part I

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Introduction

Based on: M.Hairer “A theory of regularity structures"
arXiv:1303.5113.

Introduction

Aim: Construct solutions for very singular SPDEs.

Examples:

$$\partial_t h = \partial_x^2 h + (\partial_x h)^2 + \xi, \qquad d = 1, \qquad \text{(KPZ)}$$

$$\partial_t u = \Delta u + f_{ij}(u)\,\partial_i u\,\partial_j u + g(u)\eta \qquad d = 2, \qquad \text{(PAMg)}$$

$$\partial_t \Phi = \Delta \Phi - \Phi^3 + \xi \qquad d = 3. \qquad (\Phi^4_3)$$

$\xi =$ space-time white noise, $\eta =$ spatial white noise.

(KPZ) model for growth of $1+1$-dimensional surfaces. (PAM) diffusion in a random environment. (Φ^4) model for ferro magnet near its critical temperature.

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Difficulty: Solution is too irregular to make sense of some products. $(u, v) \mapsto uv$ for $u \in \mathcal{B}^\alpha_{\infty,\infty}$, $v \in \mathcal{B}^s_{\infty,\infty}$ only well defined if $s + \alpha > 0$.

Relation to QFT

In QFT Φ^4_3 is measure μ on $\mathcal{D}'(\mathbb{R}^3)$ which formally satisfies

$$\mu(d\phi) = \frac{1}{Z} \exp\Big(- \mathcal{S}(\phi)\Big) \prod_{x \in \mathbb{R}^3} d\phi_x$$

where $\mathcal{S}(\phi) = \int |\nabla \phi|^2 + \phi^4 \, dx$.

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On $\mathbb{R}^d$ under very general assumptions on $\mathcal{S}\colon \mathbb{R}^d \to \mathbb{R}$ the measure

$$\mu = \frac{1}{Z}\exp(-\mathcal{S}(x))\,dx$$

is the unique invariant measure of the stochastic differential equation

$$dx(t) = -\nabla\mathcal{S}(x(t))\,dt + \sqrt{2}\,dW(t).$$

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SPDE can formally be rewritten

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Rigorous (at least in 1 and 2 dimensions in finite volume) by finite dimensional approximations.

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Equations of the form

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$$\partial_t \tilde{h} = \partial_x^2 \tilde{h} + \varepsilon^{1/2}(\partial_x \tilde{h})^2 + \tilde{\xi}\ .$$

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Note: Sub-criticality corresponds to superrenormalisable theories.

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Note: Different from $1/3$; $2/3$; 1-scaling. This is interesting on large scales.

Type of solution

Naive approach: Solve $\mathcal{L}u_\varepsilon = F(u_\varepsilon, \xi_\varepsilon)$ for $\xi_\varepsilon =$ smoothened noise, then remove smoothing. Does not converge as $\varepsilon \to 0$. Equations need to be renormalised.

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Examples:

$$\partial_t h_\varepsilon = \partial_x^2 h_\varepsilon + ((\partial_x h_\varepsilon)^2 - C_\varepsilon) + \xi_\varepsilon$$

$$\partial_t u_\varepsilon = \Delta u_\varepsilon + f_{ij}(u_\varepsilon)\,(\partial_i u_\varepsilon\,\partial_j u_\varepsilon - C_\varepsilon \delta_{i,j}) + g(u_\varepsilon)(\eta_\varepsilon - \hat{C}_\varepsilon g'(u_\varepsilon))$$

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Main result: There are choices of $C_\varepsilon, \hat{C}_\varepsilon$ such that solutions converge to limit which is independent of choice of mollifier.

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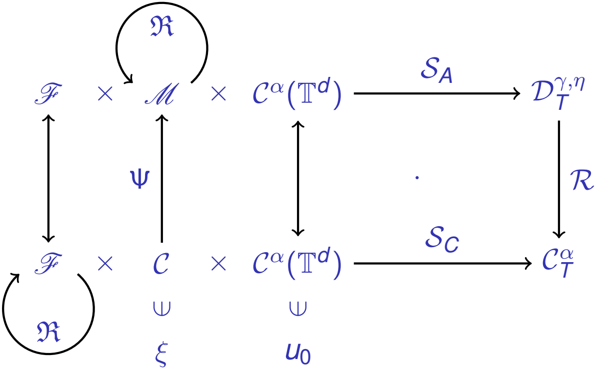
Construction gives detailed description of local structure of solutions, regularity, approximations.

Schematic construction



$\mathscr{F}$ =" {right hand sides}",

$\mathscr{M}$ =" {Models }",

$\mathcal{D}_T^{\gamma,\eta}$ =" {modelled distributions } ",

$\mathcal{R}$ ="Reconstruction operator",

$\mathfrak{R}$ ="Renormalisation group."

Canonical regularity structure I

Canonical regularity structure:

- $\hat{T} = \bigoplus_{\alpha \in \mathbb{N}_0} T_\alpha = \mathbb{R}[X_1, \ldots, X_d]$
  where $T_\alpha = \{$ homogeneous polynomials of degree α $\}$.
- $\hat{G} = \mathbb{R}^d$ acts on $\hat{T}$ by translation. For $h \in \mathbb{R}^d$ define
  $\Gamma_h P(X) = P(X - h)$.

For $a \in T_\alpha$ we have $\Gamma_h a - a \in \oplus_{\beta < \alpha} T_\beta$.

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Clearly $\Pi_y \tau(z) = \Gamma_{y-x} \Pi_x(z)$ and for $\tau \in T_\alpha$ we have
$|\Pi_x \tau(y)| \lesssim |\tau| \, |y - x|^\alpha$.

Canonical regularity structure II

Description of functions $f\colon \mathbb{R}^d \to \mathbb{R}$ smooth.

Then f can be “lifted" to a function $F\colon \mathbb{R}^d \to T$

$$F_\alpha(x) := \sum_{|k|=\alpha} \frac{1}{k!} \partial^k f(x)\, X^k \qquad \text{for } \alpha \leq n.$$

Characterisation of Hölder spaces: $f \in \mathcal{C}^\gamma(\mathbb{R}^d)$ $\gamma \notin \mathbb{N}$ if and only if for $\alpha < \gamma$ (locally uniformly around x)

$$|F_\alpha(y) - (\Gamma_{y-x} F)_\alpha(x)| \lesssim |y - x|^{\gamma-\alpha}.$$

Example: Controlled rough paths I

Aim: Moving from perturbative to non-perturbative. Given an irregular function $X\colon [0,1] \to \mathbb{R}^d$ (say in $\mathcal{C}^\gamma$ for $0 < \gamma < 1$) for which certain non-linear operations are defined, we can hope to make sense of these operations for $Y\colon [0,1] \to \mathbb{R}^d$ if “it looks like X on small scales".

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Definition (Gubinelli): Y is controlled by X if there exists a map $Y'\colon [0,1] \to \mathbb{R}^{d\times d}$ in $\mathcal{C}^\gamma$ such that for all $x, y \in [0,1]$

$$Y(y) - Y(x) = Y'(x)(X(y) - X(x)) + R_Y(y,x),$$

where $|R(x,y)| \lesssim |x-y|^{2\gamma}$.

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where $|R(x,y)| \lesssim |x-y|^{2\gamma}$.
Allows to “treat the $\mathcal{C}^\gamma$ function Y like a $\mathcal{C}^{2\gamma}$ function.

Controlled rough paths II

Regularity structure: $T = T_0 \oplus T_\gamma = \mathbb{R}^d \oplus \mathbb{R}^{d\times d}$. Group $G = \mathbb{R}^d$ acting as $\Gamma(a, b) = (a - bh, b)$.

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Model: $(a,b) \in T$, $x, y \in [0,1]$

$$\Pi_x(a,b) = a + b(X(y) - X(x))$$

$$\Gamma_{x,y}(a,b) = (a + b(X(y) - X(x)), b).$$

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Functions: $[0,1] \mapsto T = \mathbb{R}^d \oplus \mathbb{R}^{d\times d}$ is controlled by X if

$$|Y_0(y) - (\Gamma_{x,y} Y(x))_0| \lesssim |y-x|^{2\gamma}$$
$$|Y_\gamma(y) - (\Gamma_{x,y} Y(x))_\gamma| \lesssim |y-x|^{\gamma}.$$

Definition of regularity structure

A *regularity structure* $\mathbb{T} = (A, T, G)$ consists of

- An index set $A \subset \mathbb{R}$. We want $0 \in A$, A bounded from below, and A locally finite.

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- A *structure group* G of linear operators acting on T. For $\Gamma \in G$ and $a \in T_\alpha$, one has

$$\Gamma a - a \in \bigoplus_{\beta < \alpha} T_\beta \; .$$

Example: Polynomials on $\mathbb{R}^d$: $A = \mathbb{N}_0$,
$T_\alpha = \{a_k X^k, |k| = \alpha\}$ homogeneous polynomials of degree α
$G \approx \mathbb{R}^d$, $\Gamma_h P(X) = P(X - h)$.

Definition of model

A *model* for $\mathbb{T} = (A, T, G)$ on $\mathbb{R}^d$ consists of:

- A map $\Gamma\colon \mathbb{R}^d \times \mathbb{R}^d \to G$ such that $\Gamma_{xx} = \mathrm{Id}$, and such that $\Gamma_{xy}\,\Gamma_{yz} = \Gamma_{xz}$ for all x, y, z.
- Continuous linear maps $\Pi_x\colon T \to \mathcal{S}'(\mathbb{R}^d)$ such that $\Pi_y = \Pi_x \circ \Gamma_{xy}$ for all x, y.

Example: *Controlled rough paths:* $A = \{0, \alpha\}$, $T = \mathbb{R}^d \oplus \mathbb{R}^{d\times d}$, $G = \mathbb{R}^d$.
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Locally uniformly in x, y

$$|(\Pi_x a)(\mathcal{S}_x^\delta \phi)| \lesssim \|a\| \delta^\ell\;, \quad \|\Gamma_{xy} a\|_m \lesssim \|a\|\, \|x - y\|^{\ell - m}\;, .$$

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Definition of modelled distribution

Fix a regularity structure $\mathbb{T}$ and a model (Π, Γ). Then, for $\gamma \in \mathbb{R}$, the space of *modelled distributions* $\mathcal{D}^\gamma$ consists of all T_γ^--valued functions f such that

$$|||f|||_{\gamma;\mathfrak{K}} = \sup_x \sup_{\beta<\gamma} \|f(x)\|_\beta + \sup_{\|x-y\|_\mathfrak{s}\leq 1} \sup_{\beta<\gamma} \frac{\|f(x) - \Gamma_{xy} f(y)\|_\beta}{\|x-y\|_\mathfrak{s}^{\gamma-\beta}} < \infty \;.$$

Example: $\mathcal{C}^\gamma$-functions, Controlled distributions.

Reconstruction Theorem

$\mathbb{T} = (A, T, G) =$ regularity structure, $(\Pi, \Gamma) =$ model $\alpha = \min A$.

For every $\gamma > 0$, there exists a unique, continuous linear map $\mathcal{R}\colon \mathcal{D}^\gamma \to \mathcal{C}^\alpha$ such that (locally uniformly)

$$\big|(\mathcal{R}f - \Pi_x f(x))(\mathcal{S}_x^\delta \eta)\big| \lesssim \delta^\gamma \|\Pi\|_\gamma \, \|\!|f|\!\|_\gamma \;.$$

$\mathcal{S}_x^\delta \eta =$ scaled testfunction.

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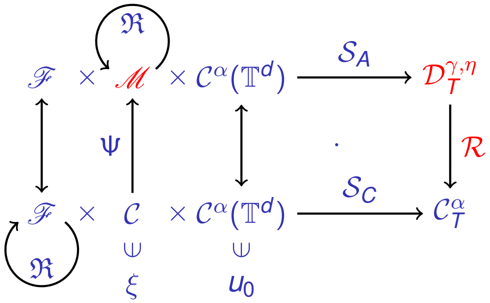
The condition $\gamma > 0$ corresponds to the condition $s + \alpha > 0$ in the multiplicative inequality for $u \in \mathcal{B}^\alpha_{\infty,\infty}$ and $v \in \mathcal{B}^s_{\infty,\infty}$.

Where are we?



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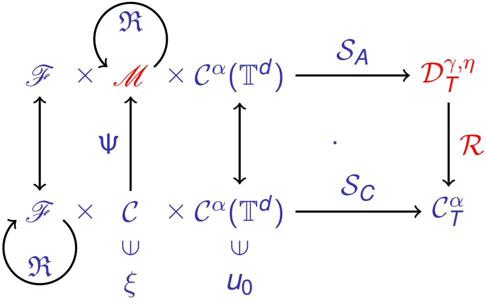

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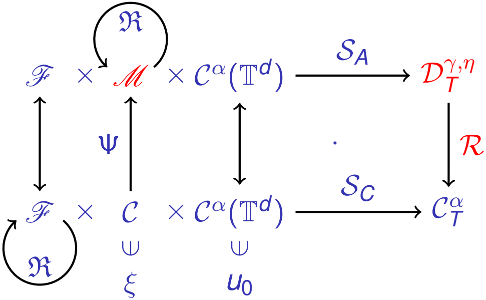

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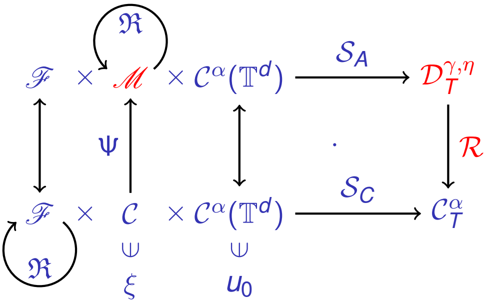

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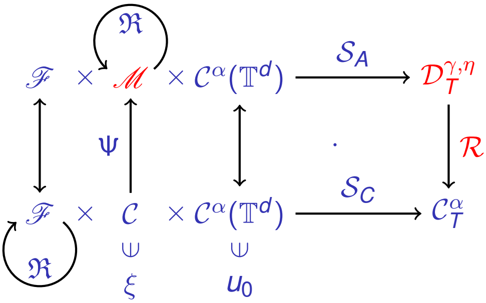

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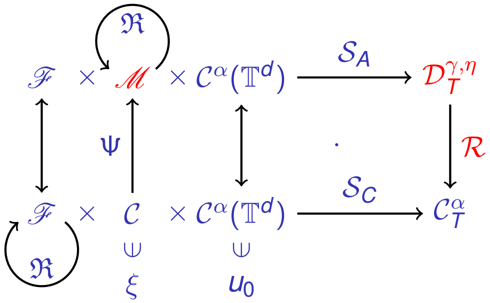

Missing:

- Need to define operations for modelled distributions:
  - ∗ Multiplication.
  - ∗ Integration against heat kernel.
- Need to build the right regularity structure for a given model.
- Need to check that renormalised models converge.