Renormalization group for two Hopf algebras in semi-direct product

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Algebra

An unital algebra is a triplet $(\mathcal{A}, m, u)$ where $\mathcal{A}$ is a k-vector space and $m : \mathcal{A} \otimes \mathcal{A} \longrightarrow \mathcal{A}$, $u : k \longrightarrow \mathcal{A}$ are two linear maps satisfying the following two axioms :

- Associativity

$$m \circ (m \otimes Id) = m \circ (Id \otimes m)$$

- unity

$$m \circ (u \otimes Id) = Id = m \circ (Id \otimes u)$$

coalgebra

A co-unital coalgebra is a triplet $(\mathcal{C}\,;\Delta\,;\varepsilon)$ where $\mathcal{C}$ is a k-vector space and $\Delta : \mathcal{C} \longrightarrow \mathcal{C} \otimes \mathcal{C}$, $\varepsilon : \mathcal{C} \longrightarrow k$ are two linear maps satisfying the following two axioms :

- Co-associativity

$$(\Delta \otimes Id) \circ \Delta = (Id \otimes \Delta) \circ \Delta$$

- co-unit

$$(\varepsilon \otimes Id) \circ \Delta = Id_{\mathcal{C}} = (Id \otimes \varepsilon) \circ \Delta.$$

Sweedler's notation

$$\Delta(x) = \sum_{(x)} x_1 \otimes x_2.$$

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Bialgebra

A bialgebra is a familly $(\mathcal{H}, m, u, \Delta, \varepsilon)$ such that :

- $(\mathcal{H}, m, u)$ is an unital algebra.
- $(\mathcal{H}, \Delta, \varepsilon)$ is a co-unital coalgebra.
- Δ and ε are morphisms of unital algebras.

Convolution product

Definition

Let $\mathcal{C}$ be a coalgebra and $\mathcal{A}$ be an algebra. Then there is an associative product on $Hom(\mathcal{C}, \mathcal{A})$ called the convolution product. It is given for $f, g \in Hom(\mathcal{C}, \mathcal{A})$ by :

$$f * g = m \circ (f \otimes g) \circ \Delta.$$

Sweedler's notation : for all $x \in \mathcal{C}$:

$$f * g(x) = \sum_{(x)} f(x_1)g(x_2).$$

Hopf algebra

Definition

A Hopf algebra is a bialgebra $\mathcal{H}$ together with a linear map $S : \mathcal{H} \longrightarrow \mathcal{H}$ which is the inverse of the identity $Id_{\mathcal{H}}$ for the convolution product on $Hom(\mathcal{H}, \mathcal{H})$.

Sweedler's notation : for all $x \in \mathcal{H}$:

$$\sum_{(x)} S(x_1)x_2 = \sum_{(x)} x_1 S(x_2) = (u \circ \varepsilon)(x).$$

Graduation

A graded Hopf algebra on k is a graded k-vector space :

$$\mathcal{H} = \bigoplus_{n \geq 0} \mathcal{H}_n$$

endowed with a product $m : \mathcal{H} \otimes \mathcal{H} \longrightarrow \mathcal{H}$, a coproduct $\Delta : \mathcal{H} \longrightarrow \mathcal{H} \otimes \mathcal{H}$ and an antipode $S : \mathcal{H} \longrightarrow \mathcal{H}$ such that :

$$m(\mathcal{H}_p \otimes \mathcal{H}_q) \subset \mathcal{H}_{p+q},$$

$$\Delta(\mathcal{H}_n) \subset \bigoplus_{p+q=n} \mathcal{H}_p \otimes \mathcal{H}_q,$$

$$S(\mathcal{H}_n) \subset \mathcal{H}_n.$$

Connected Hopf algebra

A graded Hopf algebra $\mathcal{H}$ on k is connected if :

$$\dim \mathcal{H}_0 = 1$$

Sweedler Notation : For all $x \in \mathcal{H}_n$ we have :

$$\Delta x = x \otimes \mathbf{1} + \mathbf{1} \otimes x + \sum_{(x)} x' \otimes x''$$

$$S(x) = -x - \sum_{(x)} S(x')x''$$

where : $1 \leq |x'|; |x''| \leq n-1.$

Renormalisation scheme

Let $\mathcal{H}$ be a connected graded Hopf algebra and $\varphi$is a character of $\mathcal{H}$ with values in a commutative unital algebra $\mathcal{A}$. We consider the situation where the algebra $\mathcal{A}$ admits a renormalization scheme, i.e. a splitting into two subalgebras :

$$\mathcal{A} = \mathcal{A}_- \oplus \mathcal{A}_+,$$

where $\mathcal{A}_-$ and $\mathcal{A}_+$ are two subalgebra of $\mathcal{A}$ and $\mathbf{1}_{\mathcal{A}} \in \mathcal{A}_+$

Minimal scheme :

$$\mathcal{A} = \mathcal{A}_- \oplus \mathcal{A}_+$$

where : $\mathcal{A} := \mathbb{C}[z^{-1}, z]]$, $\mathcal{A}_+ := \mathbb{C}[[z]]$ and $\mathcal{A}_- := z^{-1}\mathbb{C}[z^{-1}]$.

Proposition

The space of characters of $\mathcal{H}$ with values in $\mathcal{A}$ is a group for the convolution product, denoted by $G_{\mathcal{A}}$ and let φ be a character of $\mathcal{H}$.

The unit element e is given by $e(\mathbf{1}) = \mathbf{1}_{\mathcal{A}}$ and $e(x) = 0$ if x is homogeneous of degree ≥ 1.

The inverse is given by :

$$\varphi^{*-1} = \varphi \circ S.$$

Theorem

- Any character $\varphi \in G_{\mathcal{A}}$ admits a unique Birkhoff decomposition :

$$\varphi = \varphi_-^{*-1} * \varphi_+$$

compatible with the renormalization scheme.

- Let π be the projection on $\mathcal{A}_-$ parallel to $\mathcal{A}_+$.

$$\varphi_-(x) = -\pi\left(\varphi(x) + \sum_{(x)} \varphi_-(x')\varphi(x'')\right),$$

$$\varphi_+(x) = (I - \pi)\left(\varphi(x) + \sum_{(x)} \varphi_-(x')\varphi(x'')\right),$$

Definition

Let $\mathcal{H}$ be a connected graded Hopf algebra. The grading induces a biderivation $Y : \mathcal{H} \to \mathcal{H}$ defined for any homogeneous element of degree n by :

$$Y(x) = nx$$

We deduce an action of $\mathbb{C}$ on a group of characters $G_{\mathcal{A}}$ given by :

$$\varphi_t(x)(z) = e^{tz|x|}\varphi(x)(z).$$

Definition

Let $\mathcal{H}$ be a commutative connected Hopf algebra. The Dynkin operator is the linear endomorphism D of $\mathcal{H}$ defined by :

$$D = S * Y.$$

The set of local characters is defined by :

$$G_{\mathcal{A}}^{loc} = \{\varphi \in G_{\mathcal{A}} \text{ such that } \frac{d}{dt}(\varphi_t)_- = 0\}.$$

The renormalization group of a local character φ is defined by :

$$F_t(\varphi)(x) = \lim_{z \longrightarrow 0} (\varphi^{*-1} * \varphi_t)(x)(z).$$

The Beta function is the generator of the one-parameter group :

$$\beta(\varphi)(x) := \frac{d}{dt}\Big|_{t=0} F_t(\varphi)(x).$$

We consider two interacting connected graded Hopf algebras $\mathcal{H}$ and $\mathcal{K}$, the former being a comodule-coalgebra on the latter. we assume that there exists a coaction $\Phi : \mathcal{H} \longrightarrow \mathcal{K} \otimes \mathcal{H}$ such that the following diagram commutes :

$$\begin{array}{ccc}
\mathcal{H} & \xrightarrow{\Phi} & \mathcal{K} \otimes \mathcal{H} \\
\downarrow \Delta & & \\
\mathcal{H} \otimes \mathcal{H} & & \downarrow I \otimes \Delta \\
\downarrow \Phi \otimes \Phi & & \\
\mathcal{K} \otimes \mathcal{H} \otimes \mathcal{K} \otimes \mathcal{H} & \xrightarrow{m^{1,3}} & \mathcal{K} \otimes \mathcal{H} \otimes \mathcal{H}
\end{array}$$

where $m^{1,3} : \mathcal{K} \otimes \mathcal{H} \otimes \mathcal{K} \otimes \mathcal{H} \longrightarrow \mathcal{K} \otimes \mathcal{H} \otimes \mathcal{H}$ is defined by :

$$m^{1,3}(a \otimes b \otimes c \otimes d) = ac \otimes b \otimes d,$$

In Sweedler's notation the coaction Φ is given for all $x \in \mathcal{H}$ by :

$$\Phi(x) = \sum_{(x)} x_0 \otimes x_1 = \mathbf{1}_{\mathcal{K}} \otimes x + \sum_{(x)} x^{(')} \otimes x^{('')},$$

where : $1 \leq \left| x^{('')} \right| \leq \mid x \mid - 1$ and $\left| x^{('')} \right| + deg\ x^{(')} = \mid x \mid$,

where $|...|$ denotes the degree in $\mathcal{H}$ and *deg* denotes the degree in $\mathcal{K}$.

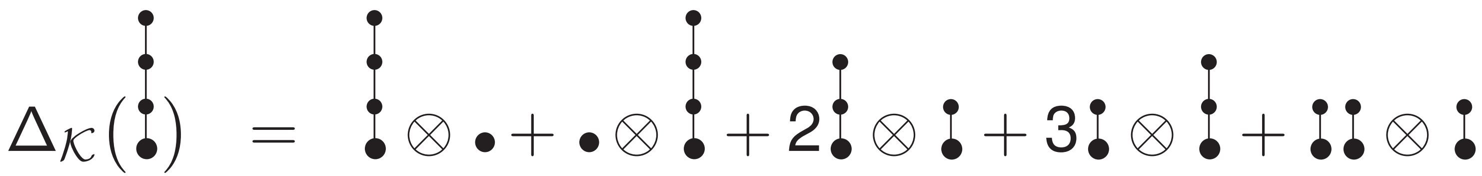
D. Calaque, K. Ebrahimi-fard and D. Manchon studied the Connes-Kreimer Hopf algebra of rooted trees $\mathcal{H}$ graded by the number of vertices, as comodule over a Hopf algebra $\mathcal{K}$ of rooted trees graded by the number of internal edges. This structure is defined as follows : For any non-empty tree t :

$$\Phi(t) = \Delta_{\mathcal{K}}(t) = \sum_{s \subseteq t} s \otimes t/s,$$

We can write $\Phi(t)$ in the following way :

$$\Phi(t) = \Delta_{\mathcal{K}}(t) = \sum_{s \subseteq t} s \otimes t/s$$

$$= \bullet \otimes t + \left(t \otimes \bullet + \sum_{s \text{ proper sub-forest of } t} s \otimes t/s \right),$$



The coproduct of Connes-Kreimer $\Delta_{\mathcal{H}}$ is defined by :

$$\Delta_{\mathcal{H}}(t) = t \otimes \mathbf{1} + \mathbf{1} \otimes t + \sum_{c \in Adm(t)} P^c(t) \otimes R^c(t),$$

where $Adm(t)$ denotes the set of admissible cuts of the forest t, $R^c(t)$ is the connected component of the root of t after the cut, and $P^c(t)$ is the forest formed by the remaining components.

Theorem

The following identity is satisfied :

$$(Id_{\mathcal{K}} \otimes \Delta_{\mathcal{H}}) \circ \Phi = m^{1,3} \circ (\Phi \otimes \Phi) \circ \Delta_{\mathcal{H}},$$

Definitions

Let Γ be an oriented Feynman graph, let $\mathcal{V}(\Gamma)$ be the set of its vertices and let P be a non-empty subset of $\mathcal{V}(\Gamma)$.

The subgraph $\Gamma(P)$ associated to P is defined as follows :

- The internal edges of $\Gamma(P)$ are the internal edges of Γ with source and target in P.
- The external edges are the external edges of Γ with source or target in P, as well as the internal edges of Γ with one end in P and the other end outside P.

A covering subgraph of Γ is a oriented Feynman graph γ, given by a collection $\{\Gamma(P_1), \cdots, \Gamma(P_n)\}$ of connected subgraphs such that $P_j \cap P_k = \emptyset$ for $j \neq k$.

The oriented cycle-free graphs generate both a Hopf algebra $\mathcal{H}$ (graded by number of vertices) and a bialgebra $\tilde{\mathcal{K}}$ (graded by number of internal edges). The left coaction of $\tilde{\mathcal{K}}$ on $\mathcal{H}$ is defined by :

$$\tilde{\Phi}(\mathbf{1}_{\mathcal{H}}) = \mathbf{1}_{\tilde{\mathcal{K}}} \otimes \mathbf{1}_{\mathcal{H}},$$

and for any non-empty graph Γ by :

$$\tilde{\Phi}(\Gamma) = \Delta_{\tilde{\mathcal{K}}}(\Gamma) = \sum_{\substack{\gamma \text{ covering subgraph } \Gamma \\ \text{compatible with the partial order}}} \gamma \otimes \Gamma/\gamma.$$

The coproduct of $\mathcal{H}$ is defined for every cycle-free oriented Feynman graph Γ by :

$$\Delta_{\mathcal{H}}(\Gamma) = \sum_{V_1 \cup V_2 = \mathcal{V}(\Gamma),\, V_1 \prec V_2} \Gamma(V_1) \otimes \Gamma(V_2), \qquad (1)$$

where the inequality $V_1 \prec V_2$ means that for any comparable $v_1 \in V_1$ and $v_2 \in V_2$ we have $v_1 \prec v_2$ in the poset $\mathcal{V}(\Gamma)$

Theorem :

The coaction $\tilde{\Phi}$ verifies :

$$(Id_{\tilde{\mathcal{K}}} \otimes \Delta_{\mathcal{H}}) \circ \tilde{\Phi} = m^{1,3} \circ (\tilde{\Phi} \otimes \tilde{\Phi}) \circ \Delta_{\mathcal{H}}.$$

Let $\mathcal{K}$ be the connected graded Hopf algebra defined by :

$$\mathcal{K} = \tilde{\mathcal{K}} / \mathcal{J}$$

where $\mathcal{J}$ is the (bi-)ideal generated by the elements $\Gamma - \mathbf{1}_{\tilde{\mathcal{K}}}$ where Γ is a graph with no internal edges.

The coaction Φ deduced from $\tilde{\Phi}$ by passing to the quotient is written :

$$\Phi(\Gamma) = \mathbf{1}_{\tilde{\mathcal{K}}} \otimes \Gamma + \left(\Gamma \otimes \bullet + \sum_{\substack{\gamma \text{ covering subgraph proper to } \Gamma, \\ \text{compatible with the partial order}}} \gamma \otimes \Gamma/\gamma \right),$$

Theorem :

the coaction Φ verifies :

$$(Id_{\mathcal{K}} \otimes \Delta_{\mathcal{H}}) \circ \Phi = m^{1,3} \circ (\Phi \otimes \Phi) \circ \Delta_{\mathcal{H}}.$$

Groups of characteres

We denote by $G_{\mathcal{A}}$(resp. $G_{\mathcal{A}}^{\mathcal{K}}$) the group of characters of $\mathcal{H}$ (resp. $\mathcal{K}$) and $\mathfrak{g}_{\mathcal{A}}$, $\mathfrak{g}_{\mathcal{A}}^{\mathcal{K}}$ the Lie algebras of infinitesimal characters associated with these groups.

Definition :

Any $\alpha \in G_{\mathcal{A}}^{\mathcal{K}}$ is written : $\exp^\star X$ where $X \in \mathfrak{g}_{\mathcal{A}}^{\mathcal{K}}$. We defined the bijection Z by :

$$\begin{aligned} Z : G_{\mathcal{A}}^{\mathcal{K}} &\longrightarrow G_{\mathcal{A}}^{\mathcal{K}} \\ \exp^\star X &\longmapsto \exp^\star zX \end{aligned}$$

where $\exp^\star zX$ is defined by :

$$\exp^\star zX(x) = \sum_{n>0} \frac{z^n}{n!} X^{\star n}(x),$$

For any $g, g' \in G_{\mathcal{A}}^{\mathcal{K}}$ we put :

$$g \star_z g' := Z^{-1}(Z(g) \star Z(g')).$$

Definition :

The action of $G_{\mathcal{A}}^{\mathcal{K}}$ on $G_{\mathcal{A}}$ is defined for all $g \in G_{\mathcal{A}}^{\mathcal{K}}, \varphi \in G_{\mathcal{A}}, x \in \mathcal{H}$ and $z \in \mathbb{C}$ by :

$$(g \star_z \varphi)(x)(z) := (Z(g) \star \varphi)(x)(z).$$

Definition :

Let $\alpha : \mathcal{K} \longrightarrow \mathcal{A}_+$ be a linear map. The operator $B_\alpha : \mathcal{H} \longrightarrow \mathcal{H}$ is defined by :

$$B_\alpha = (\alpha \otimes Id_{\mathcal{H}}) \circ \Phi$$

i.e :

$$B_\alpha(x) = \sum_{(x)} < \alpha,\, x_1 > x_0 \tag{2}$$

Proposition :

If $\alpha : \mathcal{K} \longrightarrow \mathcal{A}$ is an infinitesimal charactere of $\mathcal{K}$ then the operator B_α is a biderivation of Hopf algebra $\mathcal{H}$.

Corollary :

If $\alpha : \mathcal{K} \longrightarrow \mathcal{A}$ is infinitesimal character of $\mathcal{K}$ then $\varphi \longmapsto \varphi \circ B_\alpha = \alpha \star \varphi$ is a derivation of $\mathcal{L}(\mathcal{H}, \mathcal{A})$ for the convolution product.

For all $\alpha \in \mathfrak{g}_{\mathcal{K}}^{\mathcal{A}_+}$, we obtain a one parameter group $\theta_{t,\alpha}$ of automorphisms of $G_{\mathcal{A}}$ defined for all $\varphi \in G_{\mathcal{A}}$ by :

$$\theta_{t,\alpha}(\varphi)(x)(z) = (exp^{\star} tz\alpha \star \varphi)(x)(z).$$

The last formula also defines a one parameter subgroup of automorphisms of algebra $(\mathcal{L}(\mathcal{H}, \mathcal{A}), *)$. We note :

$$\varphi_{t,\alpha} := \theta_{t,\alpha}(\varphi).$$

In terms of Birkhoff decomposition $\varphi_{t,\alpha}$ is :

$$\varphi_{t,\alpha} = (\varphi_{t,\alpha})_-^{*-1} * (\varphi_{t,\alpha})_+.$$

We note $G_\mathcal{A}^\alpha = \{\varphi \in G_\mathcal{A} \text{ such that } \frac{d}{dt}(\varphi_{t,\alpha})_- = 0.\}$

Proposition :

For all $\alpha \in \mathfrak{g}_\mathcal{K}^{\mathcal{A}_+}$, the equation :

$$\alpha \star \varphi = \varphi * \gamma$$

define an application :

$$\begin{aligned} \tilde{\mathcal{R}}_\alpha : G_\mathcal{A} &\longrightarrow \mathfrak{g}_\mathcal{A} \\ \varphi &\longmapsto \gamma \end{aligned}$$

$$\tilde{\mathcal{R}}_\alpha(\varphi) = \varphi^{*-1} * (\alpha \star \varphi)$$

For all $\alpha \in \mathfrak{g}_{\mathcal{K}}^{\mathcal{A}_+}$ we define the operator E_α by :

$$E_\alpha := S * B_\alpha,$$

where S is the antipode of $\mathcal{H}$.

Proposition :

Let $\mathcal{H}$ be a commutative connected graded Hopf algebra. Then

$$\tilde{\mathcal{R}}_\alpha(\varphi) = \varphi \circ E_\alpha$$

.

Definition

For any $\varphi \in \mathcal{L}(\mathcal{H}\mathcal{A})$, we associate a linear form $Res\varphi$ on $\mathcal{H}$ by extracting the z^{-1} term : more precisely, if we have for any $x \in \mathcal{H}$ and for any z in a neighborhood of 0 :

$$\varphi(x)(z) = \sum_{n=-N}^{+\infty} \varphi_n(x) z^n,$$

with $\varphi_n(x) \in \mathbb{C}$, then :

$$Res\varphi(x) := \varphi_{-1}(x).$$

Theorem

- For any $\varphi \in G_{\mathcal{A}}$ there is a one-parameter family $h_{t,\alpha}$ in $G_{\mathcal{A}}$ such that : $\varphi_{t,\alpha} = \varphi * h_{t,\alpha}$ and we have :

$$\dot{h}_{t,\alpha} = \frac{d}{dt} h_{t,\alpha} = h_{t,\alpha} * z\tilde{\mathcal{R}}_\alpha(h_{t,\alpha}) + z\tilde{\mathcal{R}}_\alpha(\varphi) * h_{t,\alpha}.$$

- For $\varphi \in G_{\mathcal{A}}^\alpha$, the constant term of $h_{t,\alpha}$ defined by :

$$F_{t,\alpha}(\varphi)(x) = \lim_{z \longrightarrow 0} h_{t,\alpha}(x)(z)$$

is a one-parameter subgroup of $G_{\mathcal{A}} \cap \mathcal{L}(\mathcal{H}, \mathbb{C})$.

Definition

For any $\varphi \in G^{\alpha}_{\mathcal{A}}$ the function β_α is the generator of the one-parameter group $F_{t,\alpha}$. It is the element of $\mathcal{H}^*$ defined by :

$$\beta_\alpha(\varphi) := \frac{d}{dt}\Big|_{t=0} F_{t,\alpha}(\varphi).$$

Proposition

For any $\varphi \in G^{\alpha}_{\mathcal{A}}$ the function β_α of φ coincides with the one of the negative part φ_-^{*-1} in the Birkhoff decomposition. It is given by the three expressions :

$$\begin{aligned}\beta_\alpha(\varphi) &= Res(\tilde{\mathcal{R}}_\alpha(\varphi)) \\ &= Res(\varphi_-^{*-1} \circ B_\alpha) \\ &= -Res(\varphi_- \circ B_\alpha).\end{aligned}$$

Thank you for your attention.