Renormalization and Dynamical systems

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$$\left\{\begin{array}{rcl} \dfrac{dx_1}{dt} & = & A_1(x_1,\ldots,x_\nu) \\ \vdots & & \vdots \\ \dfrac{dx_\nu}{dt} & = & A_\nu(x_1,\ldots,x_\nu) \end{array}\right.$$

ODEs and dynamical systems

Rather than computing solutions, get informations on the trajectories, using changes of coordinates : Let $\boldsymbol{x} = (x_1, \ldots, x_\nu)$ and $A \in \mathbb{C}\{\boldsymbol{x}\}^\nu$ (analytic) :

$$\frac{d\boldsymbol{x}}{dt} = A(\boldsymbol{x}) \quad \underset{\boldsymbol{x}=\varphi^{-1}(\boldsymbol{y})}{\overset{\boldsymbol{y}=\varphi(\boldsymbol{x})}{\longleftrightarrow}} \quad \frac{d\boldsymbol{y}}{dt} = B(\boldsymbol{y})$$

with

1. $\varphi(x_1, \ldots, x_\nu) = (x_1 + h.o.t, \ldots, x_\nu + h.o.t) \in \mathbb{C}\{\boldsymbol{x}\}^\nu$ an analytic **identity-tangent diffeomorphism** (group).
2. B "as simple as possible".

In this case the trajectories (solutions), as well as their behaviour, are similar. The **vector fields** A and B are analytically **conjugate.**

Remark 1. Whenever $A(0)\neq 0$, near the origin, A is conjugate to $A(0)$. This is the **Flow-box** theorem (that still work when $A \in \mathbb{C}[[\boldsymbol{x}]]$, with **formal** identity-tangent diffeomorphisms).

When $A(0) = 0$, that is the vector field A is singular the situation gets **WORSE !!!**.

Linearization

If $A(\boldsymbol{x}) = \Lambda.\boldsymbol{x} + h.o.t = (A_1, \ldots, A_\nu)$ with $\Lambda.\boldsymbol{x} = (\lambda_1 x_1, \ldots, \lambda_\nu x_\nu)$. The vector $\lambda = (\lambda_1, \ldots, \lambda_\nu)$ is called the spectrum and it seems reasonnable to see A as a **perturbation** of its linear part and try to conjugate (linearize A)

$$\frac{d\boldsymbol{x}}{dt} = A(\boldsymbol{x}) \quad \underset{\boldsymbol{x}=\varphi^{-1}(\boldsymbol{y})}{\overset{\boldsymbol{y}=\varphi(\boldsymbol{x})}{\longleftrightarrow}} \quad \frac{d\boldsymbol{y}}{dt} = \Lambda.\boldsymbol{y}$$

→ Postpone the analytic study and try to compute $\varphi = (\varphi_1, \ldots, \varphi_\nu)$:

$$\frac{d\boldsymbol{y}}{dt} = \frac{d\varphi(\boldsymbol{x})}{dt} = \sum_{i=1}^{\nu} \frac{dx_i}{dt}\frac{\partial\varphi}{\partial x_i}(\boldsymbol{x}) = \sum_{i=1}^{\nu} A_i(\boldsymbol{x})\frac{\partial\varphi}{\partial x_i}(\boldsymbol{x}) = \Lambda.\varphi(\boldsymbol{x}) = \Lambda.\boldsymbol{y}$$

→ Of there is no vector $\boldsymbol{n} = (n_1, \ldots, n_\nu)$ $(n_i \in \mathbb{N},\ |\boldsymbol{n}| = \sum n_i \geq 2)$ such that

$$\langle \boldsymbol{n}, \lambda \rangle = n_1\lambda_1 + \ldots + n_\nu\lambda_\nu = \lambda_i$$

A is a **non resonnant** vector field and φ (formal) is well–defined. Analycity remains a hard question (diophantine condition on the spectrum, Brujno's theorem ...)

→ Otherwise, A is resonant, φ is ill-defined (without even looking at analycity).

Perturbative quantum field theory (pQFT)

Start with a "Physical" model, that is an action :

$$S(\varphi) = \int_{\mathbb{R}^d} \Big(-\frac{1}{2}\varphi(x).(-\Delta + m^2)(\varphi(x)) \Big) d^d x - \boldsymbol{g} \int_{\mathbb{R}^d} \varphi^3(x) d^d x$$

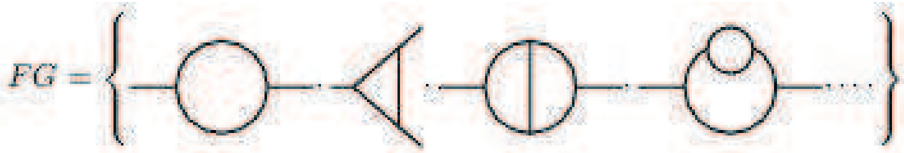
Here with have a *scalar* field $\varphi\colon \mathbb{R}^d \to \mathbb{R}$ in dimension d. After :

- Perturbative expansion with respect to $\boldsymbol{g}$,
- More or less justified integrations by part over fields,
- Fourier transforms of fields $x \to p$ (momentum) :

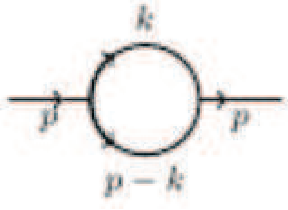
Any "observable" of the model (Energy, probabilities ...) is expressed by perturbative expansions (in $\boldsymbol{g}$) as sums and products of integrals indexed by Feynman Graphs (Feynman Integrals).

Some examples :

→ Feynman graphs :



→ Feynman integrals :



$$\xrightarrow{I_d} \quad \int \frac{1}{(k^2+m^2)((p-k)^2+m^2)} d^d k$$

Problem :

In dimension $d = 4$, some integrals $I_d(\gamma)$ are divergent (or ill-defined). Using some "tricks", we have for the previous graph :

$$I_d(\gamma) = \frac{i\pi^2}{\varepsilon} + \sum_{n \geqslant 0} a_n \varepsilon^n \; (d = 4 - \varepsilon)$$

this is Dimensional regularization.

Dimensional Regularisation :

Any Feynman integral in dimension "$4-\varepsilon$" can be computed as a Laurent series in ε and the presence of a polar part reflects the divergence of the Feynman integral in dimension $d=4$.

A naive approach : For any graph γ, substract the polar part in ε to $I_{4-\varepsilon}(\gamma)$. Do $\varepsilon=0$.
This doesn't work : It does not fit physical observations.
The right way to remove the poles is the "BPHZ" recursion, formulated by A. Connes and D. Kreimer (2000) as follows:

1. Any family $(J(\gamma))_{\gamma\in FG}$ defines an element of a group $(G_{FG},*)$.
2. For $\varphi_\varepsilon = (I_{4-\varepsilon}(\gamma))_{\gamma\in FG}$, there exists a unique factorization, the Birkhoff decomposition,

$$\varphi_\varepsilon = \varphi_\varepsilon^- * \varphi_\varepsilon^+$$

   where φ_ε^+ (resp. φ_ε^-) is regular (resp. polar) in ε.
3. When $\varepsilon=0$, the values $\varphi_0^+ = (I_4^{ren}(\gamma))_{\gamma\in FG}$ correspond to the expected renormalized values of Feynman integrals.
4. This group is a group of characters of a Hopf algebra.

	pQFT	Dynamical systems
Compute	Feynman Integrals	Coefficients of a diffeomorphism
Structure	Group of characters of a Hopf algebra	Group of diffeomorphisms
Difficulty	Divergence in some dimension	Resonant vector fields
Regularization	DimReg	???
Use the group	Birkhoff decomposition	???
Get the right result	$\varepsilon = 0$	???

But we get other tricks in dynamical systems ....

A first candidate for DimReg

Exercise 1. For $d \in \mathbb{N}$, solve $x\,y'(x) = x^d$.

Student answer : $y'(x) = x^{d-1}$ thus $y(x) = \frac{x^d}{d}$.

Correction : What about $d = 0$? The constants ?

→ Classical solution (with log):

$$y_d(x) = \begin{cases} x^d/d & (d \geq 1) \\ \log x & (d = 0) \end{cases}$$

→ Renormalization (forget log but remember powers ...) :

1. Dim. Reg. $d = \varepsilon \in \mathbb{R}^* \to y_\varepsilon(x) = x^\varepsilon/\varepsilon$
2. Solutions defined “up to a translation” (group), “Birkhoff Decomposition” at $\varepsilon = 0$:

$$y_\varepsilon^+(z) = y_\varepsilon(z) - 1/\varepsilon$$

3. $\varepsilon \to 0$: $y_0^+(x) = \log x$ works.

A toy model en dynamics :

Conjugacy problem :

$$(E_{d,\alpha})\begin{cases} \frac{dx_1}{dt} = x_1 \\ \frac{dx_2}{dt} = \alpha x_1^d x_2^2 \end{cases} \quad \overset{\Phi_d}{\sim} \quad (E_0)\begin{cases} \frac{dy_1}{dt} = y_1 \\ \frac{dy_2}{dt} = 0 \end{cases} \qquad (\alpha \in \mathbb{C},\ d \in \mathbb{N})$$

where $(x_1, x_2) = \Phi_d(y_1, y_2) = (y_1, \varphi_d(y_1, y_2))$ with

1. $\varphi_d(y_1, y_2) \in \mathbb{C}[[y_1, y_2]]$
2. Φ_d identity-tangent formal diffeomorphism.

Conjugacy equation :

$$\frac{dx_2}{dt} = \frac{d\varphi_d(y_1, y_2)}{dt} = y_1 \frac{\partial \varphi_d}{\partial y_1} + 0.\frac{\partial \varphi_d}{\partial y_2} = \alpha y_1^d \varphi_d^2 = \alpha x_1^d x_2^2$$

The equation reads : $\frac{1}{\varphi_d^2}\frac{\partial \varphi_d}{\partial y_1} = \alpha y_1^{d-1}$ and we can compute the solution.

Solution :

1. $d \in \mathbb{N}^*$:

$$(x_1, x_2) = \Phi_d(y_1, y_2) = (y_1, \varphi_d(y_1, y_2)) = \left(y_1, \frac{y_2}{1 - \alpha \frac{y_1^d}{d} y_2} \right)$$

2. Dim. Reg. : $d \in \mathbb{N}^* \rightarrow \varepsilon \in \mathbb{R}^{+*}$ (ramified powers of y_1)

$$(E_{\varepsilon,\alpha}) \overset{\Phi_\varepsilon}{\sim} (E_0)$$

3. Birkhoff Dec. : $\Phi_\varepsilon = \Phi_\varepsilon^+ \circ \Phi_\varepsilon^-$

$$\varphi_\varepsilon^-(y_1, y_2) = \frac{y_2}{1 - \frac{\alpha}{\varepsilon} y_2} \ , \ \ \varphi_\varepsilon^+(y_1, y_2) = \frac{y_2}{1 - \alpha \frac{y_1^\varepsilon - 1}{\varepsilon} y_2}$$

4. $\varepsilon \rightarrow 0$:

$$(E_{\varepsilon,\alpha}) \overset{\Phi_\varepsilon^+}{\sim} (E_0)$$

$$\lim_{\varepsilon \rightarrow 0} \Phi_\varepsilon^+(y_1, y_2) = \left(y_1, \frac{y_2}{1 - \alpha y_2 \log y_1} \right) = \Phi_0^+(y_1, y_2)$$

$$(E_{0,\alpha}) \overset{\Phi_0^+}{\sim} (E_0)$$

1. Can we generalize to :

$$(E_{d,b})\begin{cases} \frac{dx_1}{dt} = x_1 \\ \frac{dx_2}{dt} = x_1^d b(x_1,x_2) \end{cases} \text{ with } b(x_1,x_2) \in x_2^2\mathbb{C}\{x_1,x_2\} \text{ or } x_2^2\mathbb{C}[[x_1,x_2]] \text{ ?}$$

   The answer is yes but with logarithmic terms and a bit of algebra.

2. Can we have results without logarithmic terms, with the help of another dimensional regularization ? The answer is yes but :
   a) This won't conjugate $E_{0,b}$ to $E_{0,0}$ but to a "Normal form"....
   b) Once again, we will need a bit of algebra structures on ODE and diffeomorphisms....

For instance, in dimension 1, let $G^1 = \{f(x) = x + \sum_{n\geq 1} f_n x^{n+1};\, f_n \in \mathbb{C}\}$ be the group of formal identity-tangent diffeomorphisms in dimension 1 (for the composition).
If $h(x) = f \circ g(x)$, the Faà di Bruno formula gives

$$h_n = f_n + \sum_{n_0+n_1+\ldots+n_s=n} C^s_{n_0+1} f_{n_0} g_{n_1} \ldots g_{n_s} + g_n$$

This a group of characters on a Hopf algebra.

The Faà di Bruno Hopf algebra (in dimension 1)

- Let $\{X_1, X_2, ...\}$ a set of commutative variables.
- Consider the algebra $\mathcal{H}^1 = \mathbb{C}[X_1, X_2, ...]$.
- This is a Hopf algebra for the coproduct $\Delta : \mathcal{H}^1 \to \mathcal{H}^1 \otimes \mathcal{H}^1$:

$$\Delta(X_n) = X_n \otimes 1 + \sum_{n_0+n_1+...+n_s=n} C^s_{n_0+1} X_{n_0} \otimes (X_{n_1} ... X_{n_s}) + 1 \otimes X_n$$

  extended to monomials and to $\mathcal{H}^1$.
- $\mathcal{L}(\mathcal{H}^1, \mathbb{C})$ is an algebra for the product :

$$u * v = \pi_{\mathbb{C}} \circ (u \otimes v) \circ \Delta$$

- Note that any u is determined by the family of coefficients $(u(X_{n_1} ... X_{n_s}))_{s \geq 1, n_i \geq 1}$.
- If $f(x) = x + \sum_{n \geq 1} f_n x^{n+1}$. Define $u_f(X_n) = f_n$ and extend u as an algebra morphism, a character, then $u_f * u_g = u_{f \circ g}$.
- The same hold in higher dimension and the same structure arises for Feynman Graphs ...

ODEs and derivations

Let $b(x,y) \in x_2^2\mathbb{C}[[x_1,x_2]]$ and $d \in \mathbb{N}$.

$$(E_{d,b})\begin{cases} \frac{dx_1}{dt} = x_1 \\ \frac{dx_2}{dt} = x_1^d b(x_1,x_2) \end{cases} \overset{\Psi}{\sim} \quad (E_0)\begin{cases} \frac{dy_1}{dt} = y_1 \\ \frac{dy_2}{dt} = 0 \end{cases}$$

where $(y_1, y_2) = \Psi_d(x_1,x_2) = (x_1, \psi_d(x_1,x_2))$ with

1. $\psi_d(x_1,x_2) \in \mathbb{C}[[x_1,x_2]]$
2. ψ_d identity-tangent formal diffeomorphism.

The conjugacy equation reads :

$$\frac{dy_2}{dt} = \frac{dx_1}{dt}.\frac{\partial\psi_d}{\partial x_1} + \frac{dx_2}{dt}.\frac{\partial\psi_d}{\partial x_2} = x_1.\frac{\partial\psi_d}{\partial x_1} + x_1^d b(x_1,x_2).\frac{\partial\psi_d}{\partial x_2} = 0$$

More generally, if $f \in \mathbb{C}[[x_1, x_2]]$ and $\begin{cases} \frac{dx_1}{dt} = x_1 \\ \frac{dx_2}{dt} = x_1^d b(x_1, x_2) \end{cases}$, then

$$\frac{d}{dt} f(x_1, x_2) = \frac{dx_1}{dt} . \frac{\partial f}{\partial x_1} + \frac{dx_2}{dt} . \frac{\partial f}{\partial x_2} = x_1 \frac{\partial f}{\partial x_1} + x_1^d b(x_1, x_2) \frac{\partial f}{\partial x_2}$$

This gives a one-to-one correspondence such systems and differential operators :

$$\begin{cases} \frac{dx_1}{dt} = x_1 \\ \frac{dx_2}{dt} = x_1^d b(x_1, x_2) \end{cases} \quad \longleftrightarrow \quad X_b = x_1 \partial_{x_1} + x_1^d b(x_1, x_2) \partial_{x_2}$$

This will be generalized later on but, if $b(x_1,\ x_2) = \sum_{n \geqslant 0} x_1^n b_n(x_2)$, then the operator reads

$$X_b = x_1 \partial_{x_1} + x_1^d b(x_1, x_2) \partial_{x_2} = x_1 \partial_{x_1} + \sum_{n \geqslant 0} x_1^{d+n} b_n(x_2) \partial_{x_2} = x_1 \partial_{x_1} + \sum_{n \geqslant 0} x_1^{d+n} \mathbb{B}_n$$

and the conjugacy equation reads :

$$X_b . \Psi_d(x_1, x_2) = X_b . (x_1, \psi_d(x_1, x_2)) = (x_1, X_b . \psi_d(x_1, x_2)) = (x_1, 0)$$

Diffeomorphisms, substitutions automorphisms and differential operators.

We look for identity-tangent diffeomorphisms $\Phi(x_1,x_2)=(x_1,\varphi(x_1,x_2))$ with

$$\varphi\in G_{\mathfrak{A}}=\{\varphi(x_1,x_2)\in x_2+x_2^2\mathfrak{A}[[x_2]]\}\quad (\mathfrak{A}=\mathbb{C}[[x_1]])$$

Such a diffeomorphism defines a substitution automorphism on $\mathfrak{A}[[x_2]]$:

$$\forall f\in\mathfrak{A}[[x_2]],\quad F_\varphi(f)(x_2)=f\circ\varphi(x_2)$$

such that $F_\varphi(fg)=F_\varphi(f)F_\varphi(g)$. Conversely, if F is an endomorphisms on $\mathfrak{A}[[x_2]]$ such that $F(x_2)=\varphi(x_1,x_2)\in G_{\mathfrak{A}}$ and

$$\forall f,g\in\mathfrak{A}[[x_2]],\quad F(fg)=F(f)F(g)$$

then $F=F_\varphi$. Moreover, using Taylor expansions, if

$$\varphi(x_1,x_2)=x_2+\sum_{n\geq 1}\varphi_n(x_1)x_2^{n+1}\in G_{\mathfrak{A}}$$

then, using, the taylor formula $F_\varphi.f(x_2)=f(x_2+h.o.t)$, we get the differential operator

$$F_\varphi=\mathrm{Id}+\sum_{s\geq 1}\sum_{n_i\geq 1}\frac{1}{s!}\varphi_{n_1}\dots\varphi_{n_s}x_2^{n_1+\dots+n_s+s}\partial_{x_2}^s \tag{1}$$

We started with $\begin{cases} \frac{dx_1}{dt} = x_1 \\ \frac{dx_2}{dt} = x_1^d b(x_1, x_2) \end{cases}$ identified to the differential operator $X_b = x_1\partial_{x_1} + x_1^d b(x_1, x_2)\partial_{x_2}$ and we look for a diffeomorphism $\Psi(x_1, x_2) = (x_1, \psi_d(x_1, x_2))$ such that $X_b.\psi(x_1, x_2) = 0$.

But $\psi(x_1, x_2) = F_\psi(x_2)$ and the equation reads :

$$X_b.F_\psi(x_2) = (x_1\partial_{x_1} + \sum_{n\geqslant 0} x_1^{n+d}\mathbb{B}_n).F_\psi(x_2) = 0$$

with $\mathbb{B}_n = b_n(x_2)\partial_{x_2}$, $b(x_1, x_2) = \sum x_1^n b_n(x_2)$.

1. Study the equation $(x_1\partial_{x_1} + \sum_{n\geqslant 0} x_1^{n+d}\mathbb{B}_n).F_\psi.x_2 = 0$
2. Try to build F_ψ, as a differential operator, with the help of the $\mathbb{B}_n$:

$$F_\psi = \mathrm{Id} + \sum M^{n_1,\ldots,n_s}(x_1)\mathbb{B}_{n_s}\ldots\mathbb{B}_{n_1}$$

that is to say mould-comould expansions (Jean Ecalle's terminology).

Mould-Comould expansions

We have the data $X_b = x_1\partial_{x_1} + x_1^d b(x_1,\, x_2)\partial_{x_2} = x_1\partial_{x_1} + \sum_{n\geqslant 0} x_1^{n+d}\mathbb{B}_n$ ($\mathbb{B}_n = b_n(x_2)\partial_{x_2}$, $b(x_1, x_2) = \sum x_1^n b_n(x_2)$).

let

$$\mathcal{N} = \{\emptyset\} \cup \{\boldsymbol{n} = (n_1, \ldots, n_s),\ s \geq 1,\ n_i \in \mathbb{N}\}$$

and

$$\mathbb{B}_{\boldsymbol{n}} = \mathbb{B}_{n_s} \ldots \mathbb{B}_{n_1} \quad (\mathbb{B}_\emptyset = \mathrm{Id})$$

Now that we have a set of differential operators, which is called a cosymetral comould in Ecalle's work. The attempted conjugating map $(x_1,\ \psi(x_1,\ x_2))$, or rather its associated substitution automorphism, may be expressed with this comould :

$$F_\psi = \mathrm{Id} + \sum_{s\geq 1} \sum_{n_1,\ldots n_s \in \mathbb{N}} M^{n_1,\ldots,n_s}\mathbb{B}_{n_s}\ldots\mathbb{B}_{n_1} = \sum_{\boldsymbol{n}\in\mathcal{N}} M^{\boldsymbol{n}}\mathbb{B}_{\boldsymbol{n}} = \sum M^\bullet \mathbb{B}_\bullet$$

where $M^\emptyset = 1$ (for identity diffeomorphism), $F_\varphi(x_2) = \psi(x_1, x_2)$ and the collection of coefficients $M^\bullet = \{M^{\boldsymbol{n}}\}$, which is called a mould, has its values in $\mathfrak{A} = \mathbb{C}[[x_1]]$. Is there a condition that ensures that such a series is a substitution automorphism ? That is $F_\psi(fg) = F_\psi(f)F_\psi(g)$.

Reminder on moulds

Definition 2. *A mould $M^\bullet$ on $\mathcal{N}$ with values in a commutative algebra $\mathfrak{A}$ is a map from $\mathcal{N}$ to $\mathfrak{A}$. Such a mould $M^\bullet$ is symetral if $M^\emptyset = 1$ and*

$$\forall \boldsymbol{n}^1, \boldsymbol{n}^2 \in \mathcal{N}, \quad M^{\boldsymbol{n}^1} M^{\boldsymbol{n}^2} = \sum_{\boldsymbol{n} \in \mathrm{sh}(\boldsymbol{n}^1, \boldsymbol{n}^2)} M^{\boldsymbol{n}}$$

where the sum is over all the possible shuffling of the sequences $\boldsymbol{n}^1$ and $\boldsymbol{n}^2$. A mould $M^\bullet$ is alternal if $M^\emptyset = 0$ and $\forall \boldsymbol{n}^1, \boldsymbol{n}^2 \in \mathcal{N}$, $\quad \sum_{\boldsymbol{n} \in \mathrm{sh}(\boldsymbol{n}^1, \boldsymbol{n}^2)} M^{\boldsymbol{n}} = 0$.

If the series makes sense, to any mould $M^\bullet$ one can associate an operator

$$\mathbb{M} = \sum_{\boldsymbol{n} \in \mathcal{N}} M^{\boldsymbol{n}} \mathbb{B}_{\boldsymbol{n}} = \sum M^\bullet \mathbb{B}_\bullet$$

For example,

$$x^d b(x_1, x_2) \partial_{x_2} = \sum_n x_1^{n+d} \mathbb{B}_n = \sum_{\boldsymbol{n} \in \mathcal{N}} I_d^{\boldsymbol{n}} \mathbb{B}_{\boldsymbol{n}} = \sum I_d^\bullet \mathbb{B}_\bullet$$

where $I_d^\emptyset = 0$ and $I_d^{n_1, \ldots, n_s} = \begin{cases} x^{n_1+d} & \text{if } s = 1 \\ 0 & \text{otherwise} \end{cases}$ defines an alternal mould.

If $M^\bullet$ and $N^\bullet$ are two moulds, then

$$\begin{aligned}\mathbb{M}.\mathbb{N} &= \Big(\sum_{\boldsymbol{n}^1\in\mathcal{N}} M^{\boldsymbol{n}^1}\mathbb{B}_{\boldsymbol{n}^1}\Big).\Big(\sum_{\boldsymbol{n}^2\in\mathcal{N}} N^{\boldsymbol{n}^2}\mathbb{B}_{\boldsymbol{n}^2}\Big) = \sum_{\boldsymbol{n}^1,\boldsymbol{n}^2} M^{\boldsymbol{n}^1}N^{\boldsymbol{n}^2}\mathbb{B}_{\boldsymbol{n}^1}\mathbb{B}_{\boldsymbol{n}^2}\\ &= \sum_{\boldsymbol{n}^1,\boldsymbol{n}^2} M^{\boldsymbol{n}^1}N^{\boldsymbol{n}^2}\mathbb{B}_{\boldsymbol{n}^2\boldsymbol{n}^1}\\ &= \sum_{\boldsymbol{n}}\Big(\sum_{\boldsymbol{n}^1\boldsymbol{n}^2=\boldsymbol{n}} N^{\boldsymbol{n}^1}M^{\boldsymbol{n}^2}\Big)\mathbb{B}_{\boldsymbol{n}}\end{aligned}$$

where the sum is over pairs $(\boldsymbol{n}^1, \boldsymbol{n}^2)$ whose concatenation gives $\boldsymbol{n}$. These formulas define a product on moulds :

Proposition 3. *For any moulds $M^\bullet$ and $N^\bullet$, their product $P^\bullet = M^\bullet \times N^\bullet$ is defined by*

$$\forall \boldsymbol{n}\in\mathcal{N}, \quad P^{\boldsymbol{n}} = \sum_{\boldsymbol{n}^1\boldsymbol{n}^2=\boldsymbol{n}} M^{\boldsymbol{n}^1}N^{\boldsymbol{n}^2}$$

Moreover the set of symetral moulds, is a group whose unit $1^\bullet$ is given by $1^\emptyset = 1$ and $1^{\boldsymbol{n}} = 0$ otherwise. The inverse $N^\bullet$ of a given symetral mould $M^\bullet$ is given by $N^\emptyset = 1$ and

$$N^{n_1,\dots,n_s} = (-1)^s M^{n_s,\dots,n_1}$$

Proposition 4. *If $M^\bullet$ is a symetral mould, then its associated mould-comould expansion $\mathbb{M}$ is a substitution automorphism corresponding to the diffeomorphism $m(x_1, x_2) = \mathbb{M}(x_2)$. Moreover if $M^\bullet$ and $N^\bullet$ are two symetral moulds corresponding to diffeomorphisms m and n, then the mould $P^\bullet = M^\bullet \times N^\bullet$ corresponds to the diffeomorphism $m(x_1, n(x_1, x_2))$.*

For the first part of this proposition, look at the action of $\mathbb{B}_n$:

$$\begin{aligned} \mathbb{B}_{n_1}(fg) &= \mathbb{B}_{n_1}(f)g + f\mathbb{B}_{n_1}(g) \\ \mathbb{B}_{n_1,n_2}(fg) &= \mathbb{B}_{n_1,n_2}(f)g + \mathbb{B}_{n_1}(f)\mathbb{B}_{n_2}(g) + \mathbb{B}_{n_2}(f)\mathbb{B}_{n_1}(g) + f\mathbb{B}_{n_1,n_2}(g) \end{aligned}$$

For the second part,

$$m \circ n(x, y) = \mathbb{N}.\mathbb{M}.y = \sum P^\bullet \mathbb{B}_\bullet y = \mathbb{P}.y$$

The case $d \in \mathbb{N}^*$

Suppose that $(E_{d,b})\begin{cases} \frac{dx_1}{dt} = x_1 \\ \frac{dx_2}{dt} = x_1^d b(x_1,x_2) \end{cases}$ is conjugate to $(E_0)\begin{cases} \frac{dy_1}{dt} = y_1 \\ \frac{dy_2}{dt} = 0 \end{cases}$

where $(y_1, y_2) = \Psi_d(x_1, x_2) = (x_1, \psi_d(x_1, x_2))$. We expect that

$$\psi_d(x_1, x_2) = \sum V_d^\bullet \mathbb{B}_\bullet(x_2) = \mathbb{V}_d.(x_2)$$

and, if $X_b = x_1\partial_{x_1} + b(x_1, x_2)\partial_{x_2} = x_1\partial_{x_1} + \sum_{n \geqslant 0} x_1^{n+d}\mathbb{B}_n$, the conjugacy equation reads

$$X_b.\mathbb{V}_d(x_2) = (x_1\partial_{x_1} + \sum_{n \geqslant 0} I_d^\bullet \mathbb{B}_\bullet).\Big(\sum V_d^\bullet \mathbb{B}_\bullet\Big)(x_2) = 0$$

where $V_d^\bullet$ is a symetral mould. The equation yields,

$$\begin{aligned} X_b.\mathbb{V}_d(x_2) &= (x_1\partial_{x_1} + \sum I_d^\bullet \mathbb{B}_\bullet).\Big(\sum V_d^\bullet \mathbb{B}_\bullet\Big)(x_2) &= 0 \\ &= \Big(\sum (x_1\partial_{x_1} V_d^\bullet)\mathbb{B}_\bullet + \sum (V_d^\bullet \times I_d^\bullet)\mathbb{B}_\bullet\Big)(x_2) &= 0 \end{aligned}$$

Thus we look for a symetral mould $V_d^\bullet$ such that $V_d^\emptyset = 1$ and

$$x_1\partial_{x_1} V_d^\bullet = -V_d^\bullet \times I_d^\bullet$$

We set $V_d^{\emptyset} = 1$ and, for instance :

$$x_1 \partial_{x_1} V_d^{n_1} = -\,(V_d^{\bullet} \times I_d^{\bullet})^{n_1} = -\,V_d^{\emptyset} \times I_d^{n_1} - V_d^{n_1} \times I_d^{\emptyset} = -\,x_1^{n_1+d}$$

$$x_1 \partial_{x_1} V_d^{n_1,n_2} = -\,(V_d^{\bullet} \times I_d^{\bullet})^{n_1,n_2} = -\,V_d^{n_1} I_d^{n_2} = -\,V_d^{n_1} x_1^{n_2+d}$$

A straightforward computation shows that one can make the following choice :

Proposition 5. *For $d \geq 1$, the moulds defined for $(n_1, \ldots, n_s) \in \mathcal{N}$ by*

$$U_d^{n_1,\ldots,n_s} = \frac{x_1^{n_1+\ldots+n_s+s\,d}}{(\hat{n}_1 + s\,d)(\hat{n}_2 + (s-1)d)\ldots(\hat{n}_s + d)} \quad (\hat{n}_i = n_i + \ldots + n_s)$$

$$V_d^{n_1,\ldots,n_s} = \frac{(-1)^s\, x_1^{n_1+\ldots+n_s+s\,d}}{(\check{n}_1 + d)(\check{n}_2 + 2d)\ldots(\check{n}_s + s\,d)} \quad (\check{n}_i = n_1 + \ldots + n_i)$$

are symetral and solutions of the conjugacy problem : the substitution automorphism defined by $V_d^{\bullet}$ (resp. $U_d^{\bullet}$) conjugates $(E_{b,d})$ to (E_0) (resp. (E_0) to $(E_{b,d})$).

If $d = 0$, the mould $V_d^{\bullet}$ is **ill-defined !** (for example if $n_1 = 0$). This really looks like the situation that occurs in quantum field theory and calls for some renormalization. We will now describe a renormalization scheme at $d = 0$.

Renormalization in a shuffle Hopf algebra

The shuffle Hopf algebra $\mathrm{sh}_{\mathcal{N}}$

Once again, let

$$\mathcal{N} = \{\emptyset\} \cup \{\boldsymbol{n} = (n_1, \ldots, n_s),\ s \geq 1,\ n_i \in \mathbb{N}\}$$

If

$$l(n_1, \ldots, n_s) = s\ (l(\emptyset) = 0) \qquad \|n_1, \ldots, n_s\| = n_1 + \ldots + n_s + s\ (\|\emptyset\| = 0)$$

then the linear span of $\mathcal{N}$ is a graded (for the graduation $\|.\|$) vector space. This space $\mathrm{sh}_{\mathcal{N}}$ turns to be a Hopf algebra :

Product. $\emptyset$ is the unit, for $\boldsymbol{n}^1$ and $\boldsymbol{n}^2$ in $\mathcal{N}$, the product $\pi\colon \mathrm{sh}_{\mathcal{N}} \otimes \mathrm{sh}_{\mathcal{N}} \to \mathrm{sh}_{\mathcal{N}}$ is defined by

$$m(\boldsymbol{n}^1 \otimes \boldsymbol{n}^2) = \sum_{\boldsymbol{n} \in \mathrm{sh}(\boldsymbol{n}^1, \boldsymbol{n}^2)} \boldsymbol{n}$$

Coproduct. $\Delta\emptyset = \emptyset \otimes \emptyset$. For $\boldsymbol{n} \in \mathcal{N}$,

$$\Delta(\boldsymbol{n}) = \sum_{\boldsymbol{n} = \boldsymbol{n}^1 \boldsymbol{n}^2} \boldsymbol{n}^1 \otimes \boldsymbol{n}^2$$

Examples

$$\pi((n_1)\otimes(n_2,n_3)) = (n_1,n_2,n_3) + (n_2,n_1,n_3) + (n_2,n_3,n_1)$$

$$\pi((n_1)\otimes(n_2,n_3)) = (n_1,n_2,n_3) + (n_2,n_1,n_3) + (n_2,n_3,n_1)$$

$$\Delta(n_1,n_2,n_3) = (n_1,n_2,n_3)\otimes\emptyset + (n_1,n_2)\otimes(n_3) + (n_1)\otimes(n_2,n_3) + \emptyset\otimes(n_1,n_2,n_3)$$

$\mathrm{sh}_{\mathcal{N}}$ is a very classical graded connected Hopf algebra related to moulds : by the correspondence :

$$\begin{array}{ccc} M^\bullet = \{M^{\boldsymbol{n}} \in \mathfrak{A}, \boldsymbol{n} \in \mathfrak{A}\} & \longleftrightarrow & m \in \mathcal{L}(\mathrm{sh}_{\mathcal{N}}, \mathfrak{A})\ \ (m(\boldsymbol{n}) = M^{\boldsymbol{n}}) \\ M^\bullet \times L^\bullet & \longleftrightarrow & m * \ell = \pi_{\mathfrak{A}} \circ (m \otimes \ell) \circ \Delta \\ M^\bullet \text{ symetral} & \longleftrightarrow & m \,\text{character}\, (m \circ \pi = \pi_{\mathfrak{A}} \circ (m \otimes m)) \\ M^\bullet \text{alternal} & \longleftrightarrow & m \text{ infinitesimal character} \end{array}$$

We recover the same situation as in pQFT with feynman integrals as a ill-defined character on a Hopf algebra...

Divergences for the moulds (or characters) $U_d^\bullet$ and $V_d^\bullet$.

The "character" $V_d^\bullet$ defined by

$$V_d^{n_1,\ldots,n_s} = \frac{(-1)^s\, x^{n_1+\ldots+n_s+s\,d}}{(\check{n}_1+d)(\check{n}_2+2d)\ldots(\check{n}_s+s\,d)} \quad (\check{n}_i = n_1+\ldots+n_i)$$

is ill-defined when $d=0$. When looking at $V_d^\bullet$, if

$$\forall (n_1,\ldots,n_s)\in\mathcal{N}, \quad D(n_1,\ldots,n_s) = \max\{0\le i\le s\ ;\ \forall 1\le j\le i,\ \check{n}_j=0\}$$

from the physicists point of view :

- If $D(n_1,\ldots,n_s)=0$, $n_1\neq 0$ and $V_d^{n_1,\ldots,n_s}$ has no divergence at $d=0$,
- If $D(n_1,\ldots,n_s)=1$, $n_1=0$, $\check{n}_2\neq 0$ and $V_d^{n_1,\ldots,n_s}$ has an overall divergence but no subdivergence at $d=0$,
- If $D(n_1,\ldots,n_s)>1$, $V_d^{n_1,\ldots,n_s}$ has an overall divergence and $D(n_1,\ldots,n_s)-1$ subdivergences at $d=0$.

Remember the candidate for dimensional regularisation $d=\varepsilon\in\mathbb{R}^*$. The price to pay is to consider now that

$$x^\varepsilon = \sum_{n\ge 0}\frac{\varepsilon^n}{n!}\log^n x$$

Theorem 6. $V_\varepsilon^\bullet$ *admits a unique factorisation*

$$R_\varepsilon^\bullet = C_\varepsilon^\bullet \times V_\varepsilon^\bullet$$

where $C_\varepsilon^\bullet$ *(counterterms) is a symetral mould polar in* ε *and* $R_\varepsilon^\bullet$ *(regularized) is a symetral mould regular in* ε*, but with logarithmic terms* ($\log x_1$)*. Moreover*

$$C_\varepsilon^{n_1,\ldots,n_s} = \begin{cases} \dfrac{1}{s!\varepsilon^s} & \text{if } n_1 = \ldots = n_s = 0 \\ 0 & \text{otherwise} \end{cases}$$

Examples

$$(C_\varepsilon^\bullet \times V_\varepsilon^\bullet)^{0,n} = C_\varepsilon^{0,n} + C_\varepsilon^0 V_\varepsilon^n + V_\varepsilon^{0,n} = -\frac{1}{\varepsilon}\frac{x_1^{n+\varepsilon}}{n+\varepsilon} + \frac{x_1^{n+2\varepsilon}}{\varepsilon(n+2\varepsilon)}$$

We have now a renormalization scheme for our problem but, as in quantum field theory, this would be useless if it had no meaning for our equations.

Interpretation of the renormalized mould $R_\varepsilon^\bullet$.

Ramified conjugacy.

On one hand, the ill-definedness of $V_d^\bullet$ at $d = 0$ suggest that the equation $(E_{d,b})\begin{cases} \frac{dx_1}{dt} = x_1 \\ \frac{dx_2}{dt} = x_1^d b(x_1,x_2) \end{cases}$ is not formally conjugate to $(E_0)\begin{cases} \frac{dy_1}{dt} = y_1 \\ \frac{dy_2}{dt} = 0 \end{cases}$

On the other hand, we chose a quite natural dimensional regularization for our mould $V_d^\bullet$ since for $\varepsilon \neq 0$, we still have the equation $x_1\partial_{x_1}V_\varepsilon^\bullet = -V_\varepsilon^\bullet \times I_\varepsilon^\bullet$.
As $R_\varepsilon^\bullet = C_\varepsilon^\bullet \times V_\varepsilon^\bullet$ and $C_\varepsilon^\bullet$ does not depend on x,

$$\begin{aligned} x_1\partial_{x_1}R_\varepsilon^\bullet &= x_1\partial_{x_1}(C_\varepsilon^\bullet \times V_\varepsilon^\bullet) \\ &= C_\varepsilon^\bullet \times (x_1\partial_{x_1}V_\varepsilon^\bullet) \\ &= -C_\varepsilon^\bullet \times V_\varepsilon^\bullet \times I^\bullet \\ &= \quad -R_\varepsilon^\bullet \times I^\bullet \end{aligned}$$

The mould $R_\varepsilon^\bullet$ (as $V_\varepsilon^\bullet$) defines a diffeomorphism that also conjugates the equation $(E_{\varepsilon,b})\begin{cases} \frac{dx_1}{dt} = x_1 \\ \frac{dx_2}{dt} = x_1^\varepsilon b(x_1,x_2) \end{cases}$ to $(E_0)\begin{cases} \frac{dy_1}{dt} = y_1 \\ \frac{dy_2}{dt} = 0 \end{cases}$

But the mould $R_\varepsilon^\bullet$ is regular at $\varepsilon = 0$, with a price to pay : it contains polynomials in x and $\log x$. When ε goes to 0, we get :

Theorem 7. *There exists a "ramified" identity tangent diffeomorphism* $\Psi(x_1, x_2) = (x_1,\ \psi(x_1, x_2))$ *with* $\psi(x_1, x_2) \in x_2 + x_2^2\mathbb{C}[[x_1, \log x_1, x_2]]$ *that conjugates*

$$(E_{0,b})\begin{cases} \frac{dx_1}{dt} = x_1 \\ \frac{dx_2}{dt} = b(x_1, x_2) \end{cases} \text{ to } (E_0)\begin{cases} \frac{dy_1}{dt} = y_1 \\ \frac{dy_2}{dt} = 0 \end{cases}.$$

The need for logarithms, as well as the ill-definedness of a "formal" conjugating diffeomorphism, suggest that, in the case $d = 0$, some part of the right-hand term of the equation $\frac{dx_2}{dt} = b(x_1, x_2)$ cannot be cancelled by formal conjugacy : there should remain some formal "obstructions". The next natural question becomes : If one cannot formally conjugate to (E_0), what is the most simple equation to which one can conjugate ?

The logarithmig-alogarithmic factorization of $R_0^\bullet$ and its interpretation.

Theorem 8. *The symetral mould $R_0^\bullet$ admits the following factorization :*

$$R_0^\bullet = L^\bullet \times S^\bullet$$

where

1. *$L^\bullet$ is a purely logarithmic symetral mould defined for $(n_1, \ldots, n_s) \in \mathcal{N}$ by*

$$L^{n_1,\ldots,n_s} = \begin{cases} \frac{(-1)^s}{s!}\log^s x_1 & \text{if } n_1 = \ldots = n_s = 0 \\ 0 & \text{otherwise} \end{cases}$$

2. *$S^\bullet$ is a symetral mould with values in $\mathbb{C}[[x_1]]$.*

We have then

$$\begin{aligned} x_1\partial_{x_1}R_0^\bullet &= x_1\partial_{x_1}(L^\bullet \times S^\bullet) \\ &= L^\bullet \times (x_1\partial_{x_1}S^\bullet) + (x_1\partial_{x_1}L^\bullet) \times S^\bullet \\ &= -R_0^\bullet \times I_0^\bullet \\ &= -L^\bullet \times S^\bullet \times I_0^\bullet \end{aligned}$$

and if $-L^\bullet \times A^\bullet = x_1\partial_{x_1}L^\bullet$,

$$x_1\partial_{x_1}S^\bullet + S^\bullet \times I_0^\bullet = A^\bullet \times S^\bullet$$

A straightforward computation shows that $A^\bullet$ is alternal ($A^\emptyset = 0$) and for $(n_1, \ldots, n_s) \in \mathcal{N}$,

$$A^{n_1,\ldots,n_s} = \begin{cases} 1 & \text{if } s=1 \text{ and } n_1 = 0 \\ 0 & \text{otherwise} \end{cases}$$

so that

$$\sum A^\bullet \mathbb{B}_\bullet x_2 = A^0 \mathbb{B}_0 x_2 = b(0, x_2) \quad \Big(b(x_1,x_2) = \sum x_1^n b_n(x_2),\ \ \mathbb{B}_n = b_n(x_2)\partial_{x_2}\Big)$$

but now, if φ^{nor} is the formal diffeomorphism associated to $S^\bullet$ and $(y_1,\ y_2) = \Phi^{\mathrm{nor}}(x_1,x_2) = (x_1, \varphi^{\mathrm{nor}}(x_1,x_2))$, we have

$$\frac{dy_1}{dt} = \frac{dx_1}{dt} = x_1 = y_1$$

and for $y_2 = \varphi^{\mathrm{nor}}(x_1,x_2)$,

$$\begin{aligned}
\frac{dy_2}{dt} &= (x_1\partial_{x_1} + b(x_1,x_2)\partial_{x_2})\varphi^{\mathrm{nor}}(x_1,x_2) &&= \Big(x_1\partial_{x_1} + \sum I_0^\bullet \mathbb{B}_\bullet\Big)\Big(\sum S^\bullet \mathbb{B}_\bullet\Big)x_2 \\
&= \sum (x_1\partial_{x_1} S^\bullet + S^\bullet \times I_0^\bullet)\mathbb{B}_\bullet x_2 &&= \sum (A^\bullet \times S^\bullet)\mathbb{B}_\bullet x_2 \\
&= \Big(\sum S^\bullet \mathbb{B}_\bullet\Big)\Big(\sum A^\bullet \mathbb{B}_\bullet x_2\Big) &&= \Big(\sum S^\bullet \mathbb{B}_\bullet\Big)(b(0,x_2)) \\
&= b(0, \varphi^{\mathrm{nor}}(x_1,x_2)) &&= b(0,\ y_2)
\end{aligned}$$

Normal forms

The system $\begin{cases} \frac{dx_1}{dt} = x_1 \\ \frac{dx_2}{dt} = b(x_1, x_2) \in x_2^2\mathbb{C}[[x_1, x_2]] \end{cases}$ is formally conjugate to $\begin{cases} \frac{dy_1}{dt} = y_1 \\ \frac{dy_2}{dt} = b(0, y_2) \end{cases}$.

This is a "classical" result :

1. Associate to the system $X = x_1\partial_{x_1} + b(x_1, x_2)\partial_{x_2} = X^{\text{lin}} + \mathbb{B}$ where the linear part is $X^{\text{lin}} = x_1\partial_{x_1}$.
2. A monomial $x_1^{m_1}x_2^{m_2}$ is resonant (for this linear part) if $X^{\text{lin}}.x_1^{m_1}x_2^{m_2} = 0$:

$$X^{\text{lin}}.x_1^{m_1}x_2^{m_2} = m_1 x_1^{m_1}x_2^{m_2}$$

   thus, here, any monomial $x_2^{m_2}$ is resonant.
3. Theorem : any vector field $X^{\text{lin}} + \mathbb{B}$ can be formally conjugated to a vector field $X^{\text{lin}} + \boldsymbol{N} = X^{\text{lin}} + N(x_1, x_2)\partial_{x_2}$ with $X^{\text{lin}}.N = 0$. $X^{\text{lin}} + \boldsymbol{N}$ is called a normal form (not unique) since it contains only resonant monomials ....

Back to conjugacy of formal vector fields.

In dim. ν : $(\lambda_1,\ldots,\lambda_\nu)\in\mathbb{C}^\nu$, $a=(a_1,\ldots,a_\nu), b=(b_1,\ldots b_\nu)\in(\mathbb{C}_{\geqslant 2}[[x_1,\ldots,x_\nu]])^\nu$:

$$\begin{array}{rcccl} \frac{dx_1}{dt} &=& A_1(\boldsymbol{x}) &=& \lambda_1 x_1 + a_1(\boldsymbol{x}) \\ \vdots & & & & \vdots \\ \frac{dx_\nu}{dt} &=& A_\nu(\boldsymbol{x}) &=& \lambda_\nu x_\nu + a_\nu(\boldsymbol{x}) \end{array} \quad \overset{\boldsymbol{y}=\varphi(\boldsymbol{x})}{\longleftrightarrow} \quad \begin{array}{rcccl} \frac{dy_1}{dt} &=& B_1(\boldsymbol{y}) &=& \lambda_1 y_1 + b_1(\boldsymbol{y}) \\ \vdots & & & & \vdots \\ \frac{dy_\nu}{dt} &=& B_\nu(\boldsymbol{y}) &=& \lambda_\nu y_\nu + b_\nu(\boldsymbol{y}) \end{array}$$

where $\varphi=(\varphi_1,\ldots,\varphi_\nu)=(x_1+u_1,\ldots x_\nu+u_\nu)$, $u=(u_1,\ldots,u_\nu)\in(\mathbb{C}_{\geqslant 2}[[x_1,\ldots,x_\nu]])^\nu$.

The conjugacy equation reads : $\forall\ 1\leqslant j\leqslant\nu$

$$\frac{dy_j}{dt}=\sum_{i=1}^{\nu}\frac{dx_i}{dt}\frac{\partial\varphi_j}{\partial x_i}=\sum_{i=1}^{\nu}A_i(\boldsymbol{x})\partial_{x_i}\varphi_j=B_j\circ\varphi(\boldsymbol{x})=B_j(\boldsymbol{y})$$

On $f\in\mathbb{C}[[\boldsymbol{x}]]$: $\frac{d}{dt}f(\boldsymbol{x})=\sum_{i=1}^{\nu}A_i(\boldsymbol{x})\partial_{x_i}f(\boldsymbol{x})=X_A.f$, $f\circ\varphi(x)=(F_\varphi.f)(\boldsymbol{x})$:

$$\forall 1\leqslant j\leqslant\nu,\quad X_A.F_\varphi.x_j=F_\varphi.X_B.x_j$$

$$X_A.F_\varphi=F_\varphi.X_B$$

Vector fields, derivations and homogeneous degrees

Let $\boldsymbol{n} = (n_1, \ldots, n_\nu) \in \mathbb{N}^\nu$, $|\boldsymbol{n}| = n_1 + \ldots + n_\nu$, $\boldsymbol{x}^{\boldsymbol{n}} = x_1^{n_1} \ldots x_\nu^{n_\nu}$: $\mathrm{td}(\boldsymbol{x}^{\boldsymbol{n}}) = |\boldsymbol{n}|$.

$$X_A = \sum_{i=1}^{\nu} \lambda_i x_i \partial_{x_i} + \sum_{i=1}^{\nu} a_i(\boldsymbol{x}) \partial_{x_i} = \sum_{i=1}^{\nu} \lambda_i x_i \partial_{x_i} + \sum_{i=1}^{\nu} \sum_{|\boldsymbol{n}| \geqslant 2} a_i^{\boldsymbol{n}} \boldsymbol{x}^{\boldsymbol{n}} \partial_{x_i}$$

If $\boldsymbol{n} = (n_1, \ldots, n_\nu)$ and $\boldsymbol{m} = (m_1, \ldots, m_\nu)$:

$$\lambda_i x_i \partial_{x_i} . \boldsymbol{x}^{\boldsymbol{m}} = m_i \lambda_i \boldsymbol{x}^{\boldsymbol{m}} \quad , \quad a_i^{\boldsymbol{n}} \boldsymbol{x}^{\boldsymbol{n}} \partial_{x_i} . \boldsymbol{x}^{\boldsymbol{m}} = a_i^{\boldsymbol{n}} x_1^{n_1 + m_1} \ldots x_i^{n_i + m_i - 1} \ldots x_\nu^{n_\nu + m_\nu}$$

Thus

$$\mathrm{td}(\lambda_i x_i \partial_{x_i} . \boldsymbol{x}^{\boldsymbol{m}}) = 0 + \mathrm{td}(\boldsymbol{x}^{\boldsymbol{m}}) \quad , \quad \mathrm{td}(a_i^{\boldsymbol{n}} \boldsymbol{x}^{\boldsymbol{n}} \partial_{x_i} . \boldsymbol{x}^{\boldsymbol{m}}) = |\boldsymbol{n}| - 1 + \mathrm{td}(\boldsymbol{x}^{\boldsymbol{m}}).$$

→ Homogenous degree : $\mathrm{hd}(\lambda_i x_i \partial_{x_i}) = 0$, $\mathrm{hd}(a_i^{\boldsymbol{n}} \boldsymbol{x}^{\boldsymbol{n}} \partial_{x_i}) = |\boldsymbol{n}| - 1$

→ The operator X_A decomposes in homogeneous components :

$$X_A = X_A^0 + \sum_{k \geqslant 1} X_A^k$$

→ Note that if X, Y are homogeneous derivations, then $[X, Y] = X.Y - Y.X$ is a derivation and $\mathrm{hd}([X, Y]) = \mathrm{hd}(X) + \mathrm{hd}(Y)$....

Diffeos, operators and homogeneous degrees

In dimension ν, with coordinates $\boldsymbol{x} = (x_1, \ldots, x_\nu)$, let $\varphi(\boldsymbol{x}) = (\varphi_1(\boldsymbol{x}), \ldots, \varphi_\nu(\boldsymbol{x}))$

$$\varphi_i(x) = x_i + \sum_{|\boldsymbol{n}| \geq 2} \varphi^i_{\boldsymbol{n}} \boldsymbol{x}^{\boldsymbol{n}} = x_i + u_i(\boldsymbol{x}) = x_i + \sum_{|\boldsymbol{n}| \geqslant 2} u_i^{\boldsymbol{n}} \boldsymbol{x}^{\boldsymbol{n}} \quad u_i(\boldsymbol{x}) \in \mathbb{C}_{\geqslant 2}[[x_1, \ldots, x_\nu]]$$

a formal identity–tangent diffeomorphism and $F_\varphi.f(\boldsymbol{x}) = f \circ \varphi(\boldsymbol{x}) = f(\boldsymbol{x} + u(\boldsymbol{x}))$.
The Taylor expansion of $f(\boldsymbol{x} + u(\boldsymbol{x}))$ gives:

$$F_\varphi.f(\boldsymbol{x}) = f(\boldsymbol{x} + u(\boldsymbol{x})) = f(\boldsymbol{x}) + \sum_{\substack{s \geqslant 1 \\ 1 \leqslant i_1, \ldots, i_s \leqslant \nu}} \frac{1}{s!} u_{i_1} \ldots u_{i_s} \partial_{x_{i_1}} \ldots \partial_{x_{i_s}} f(\boldsymbol{x})$$

$$F_\varphi.f(\boldsymbol{x}) = \left(\mathrm{Id} + \sum_{\substack{s \geqslant 1 \\ 1 \leqslant i_1, \ldots, i_s \leqslant \nu \\ 2 \leqslant |\boldsymbol{n}^1|, \ldots, |\boldsymbol{n}^s|}} \frac{1}{s!} u_{i_1}^{\boldsymbol{n}^1} \ldots u_{i_s}^{\boldsymbol{n}^s} \boldsymbol{x}^{\boldsymbol{n}^1 + \ldots + \boldsymbol{n}^s} \partial_{x_{i_1}} \ldots \partial_{x_{i_s}} \right) f(\boldsymbol{x})$$

and

$$\mathrm{td}\left(\frac{1}{s!} u_{i_1}^{\boldsymbol{n}^1} \ldots u_{i_s}^{\boldsymbol{n}^s} \boldsymbol{x}^{\boldsymbol{n}^1 + \ldots + \boldsymbol{n}^s} \partial_{x_{i_1}} \ldots \partial_{x_{i_s}} \boldsymbol{x}^{\boldsymbol{m}} \right) = |\boldsymbol{n}^1| + \ldots + |\boldsymbol{n}^s| - s + \mathrm{td}(\boldsymbol{x}^{\boldsymbol{m}})$$

thus

$$F_\varphi = \mathrm{Id} + \sum_{k \geq 1} F_\varphi^k \quad \mathrm{hd}(F_\varphi^k) = k$$

1. If $L = \{X = \sum_{n\geqslant 1} X_n,\, X \text{ derivation}\}$, $G = \{F_\varphi = \mathrm{Id} + \sum_{n\geqslant 1} F_n\}$ is the group of substitutions automorphisms :

$$X \in L \quad \xrightarrow{\exp} \quad F = \exp(X) = \sum \frac{X^s}{s!} \in G$$

$$X = \log(\mathrm{Id} + \tilde{F}) = \sum \frac{(-1)^{s-1}\tilde{F}^s}{s} \in L \quad \xleftarrow{\log} \quad F = \mathrm{Id} + \tilde{F} \in G$$

2. Conjugacy with $X_A = X_0 + \tilde{X}_A$, $X_B = X_0 + \tilde{X}_B$, $F = F_\varphi$:

$$(X_0 + \tilde{X}_A)F = X_A.F_\varphi = F_\varphi.X_B = F(X_0 + \tilde{X}_B)$$

thus

$$\mathrm{ad}_{X_0}(F) = [X_0, F] = F\tilde{X}_B - \tilde{X}_A F$$

3. Linearization : if $A(\boldsymbol{x}) = (\lambda_1 x_1, \ldots, \lambda_\nu x_\nu)$, then $X_A = X_0$ and X_B is linearizable if there exists F such that

$$\mathrm{ad}_{X_0}(F) = [X_0, F] = F\tilde{X}_B$$

4. Jacobi identity or $d = \mathrm{ad}_{X_0}$, $X, Y \in L$:

$$d([X,Y]) = [d(X), Y] + [X, d(Y)]$$

and if $\mathrm{hd}(X^k) = k$, then $\mathrm{hd}(d(X^k)) = k$ (preserves the “graduation”)

Generalization to (completed) graded Lie algebras

1. Polynomial vector field $A = \Lambda.\boldsymbol{x} + a$, $a \in \mathbb{C}[\boldsymbol{x}]^\nu$: $X_A = X_0 + \sum_{\text{finite}} X_n$. Let $L_0 \oplus (\oplus_{n\geqslant 1} L_n)$ a graded Lie algebra ($[L_n, L_m] \subset L_{m+n}$) and $x_0 + x$ an element ($x_0 \in L_0$, $x \in \oplus_{n\geqslant 1} L_n$).
2. Formal vector field $A = \Lambda.\boldsymbol{x} + a$, $a \in \mathbb{C}[[\boldsymbol{x}]]^\nu$: $X_A = X_0 + \sum X_n$. Let $L_0 \oplus L$ be the completed graded Lie algebra. L is the completion for the graduation of $\oplus_{n\geqslant 1} L_n$ and $x_0 + x = x_0 + \sum x_n$ an element.
3. A susbtitution automorphism $F_\psi = \mathrm{Id} + \sum F_n$ can be written $\exp(\sum X_n)$. Let $G = \exp(L)$ the Lie group associated to L and $\varphi = 1 + \sum \varphi_n \in G$. Note that $G \subset \mathcal{U}$ where $\mathcal{U}$ is the completion (for the graduation) of $\mathcal{U}(L)$.
4. For a “linear part” X_0, $d.X = \mathrm{ad}_{X_0}(X) = [X_0, X]$ preserves the graduation and is a derivation acting on the Lie algebra $\{\sum_{n\geqslant 1} X_n\}$. Let d an endomorphism on L, preserving the graduation, such the $d[x, y] = [d.x, y] + [d.x, y]$: d is a graded derivation on L, that extends to $\mathcal{U}$ (universal property +completion) with $d.1 = 0$.
5. The linearization equation $\mathrm{ad}_{X_0}(F) = F(\sum X_n)$ reads

$$d\varphi = \varphi.x \quad x \in L,\ \varphi \in G$$

Derivations and d-logarithms in Lie algebras.

Let L a completed graded Lie algebra (and $G, \mathcal{U}$ as before). A derivation d on L is a linear endomorphism on L such that $d[x,y] = [d.x, y] + [d.x, y]$. It extends to G and $\mathcal{U}$ and $d(1)=0$, $d(\mathcal{U}_n) \subset \mathcal{U}_n$, $\forall x, y \in \mathcal{U}$, $\quad d(x.y) = d(x).y + x.d(y)$.
For such a derivation, we can define and study the "differential" equation

$$d(\varphi) = \varphi.x$$

Any such derivation defines a map:

$$\begin{aligned} \log_d \ : \ 1+\mathcal{U}_{\geq 1} \ &\rightarrow \ \mathcal{U}_{\geq 1} \\ \varphi \ &\mapsto \ \varphi^{-1}.d(\varphi) = \left(1 + \sum_{s\geq 1} (-1)^s \sum_{n_1,\ldots,n_s \geq 1} \varphi_{n_1} \cdots \varphi_{n_s}\right)\left(\sum_{n \geqslant 1} d(\varphi_n)\right) \end{aligned}$$

that sends G on L (Magnus–type formula) : Let $\varphi = \exp(\alpha) \in G$ $(\alpha \in L)$, then

$$\log_d(\varphi) = \sum_{s\geq 0} \frac{(-1)^s}{(s+1)!} \mathrm{ad}_\alpha^s(d(\alpha)) = \frac{e^{-\mathrm{ad}_\alpha} - 1}{-\mathrm{ad}_\alpha}(d(\alpha))$$

where $\mathrm{ad}_\alpha(x) = [\alpha, x] = \alpha x - x\alpha$ is the adjoint action of α.

Do we have an inverse for $\log_d$ (solves the linearization problem)

The invertible case.

Theorem 9. *If d admits a graded inverse I on L, then the map $\log_d : G \to L$ is invertible. With an inverse :* $\exp_d : L \to G$.

Proof. Let $u = \sum_{n \geqslant 1} u_n \in L$ and $\varphi = \exp(v)$ with $v = \sum_{n \geqslant 1} v_n$ such that $d\varphi = \varphi u$.
thanks to the graduation and to the lemma,

$$d(v_n) + \sum_{k=1}^{n-1} \sum_{i=1}^{n-k} \frac{(-1)^i}{(i+1)!} \sum_{n_1+..+n_i=n-k} \mathrm{ad}_{v_{n_1}}(\mathrm{ad}_{v_{n_2}}...(\mathrm{ad}_{v_{n_i}}(d(v_k)))...) = u_n$$

In L_1 we have, $d(v_1) = u_1$ and we define $v_1 = I(u_1) \in L_1$.
For $n = 2$, $d(v_2) - \frac{1}{2}[v_1, d(v_1)] = u_2$ that gives :

$$v_2 = I(u_2) - \frac{1}{2} I([I(u_1), u_1])$$

and the proof follows recursively: $v = \sum_{n \geq 1} v_n$ is in L and $\varphi = \exp(v)$ is a solution of $\log_d(\varphi) = u \in L$. □

The Dynkin operator, the logarithm and other examples.

→ $d = Y : \forall n \geq 1, \forall x \in L_n, \quad Y(x) = n\, x$. This gives, for $u = \sum_{n \geqslant 1} u_n$,

$$\exp_Y(u) = 1 + \sum_{s \geqslant 1, n_i \geqslant 1} \frac{u_{n_1} \ldots u_{n_s}}{n_1(n_1 + n_2)\ldots(n_1 + \ldots + n_s)}$$

that is the Dynkin operator (or rather its inverse).

→ Let $\mu L[\mu] = \mu\mathbb{C}[\mu] \otimes L$ (L as a $\mu\mathbb{C}[\mu]$–module) and $d = \mu\partial_\mu$:

$$\exp_d(\mu u) = \exp(\mu u) \quad (d\varphi = \mu\partial_\mu\varphi = \mu\varphi u)$$

→ Let $x_0 \in L_0$, if $d = \mathrm{ad}_{x_0} = [x_0, .]$ is invertible, then $\varphi = \exp_d(u)$ is the unique solution of

$$\mathrm{ad}_{x_0}(\varphi) = \varphi.u$$

and solves the (formal) linearization problem in dynamical systems ....

Back to dynamical systems

Let $A(\boldsymbol{x}) = \Lambda.\boldsymbol{x} + a(\boldsymbol{x}) = (\lambda_1 x_1 + a_1(x), \ldots, \lambda_\nu x_\nu + a_\nu(\boldsymbol{x}))$ and $x_0 = \sum \lambda_i x_i \partial_{x_i} \in L_0$, $u = \sum a_i(\boldsymbol{x}) \partial_{x_i} \in L$. The system $\frac{d\boldsymbol{x}}{dt} = A(\boldsymbol{x})$ can be conjugated to $\frac{d\boldsymbol{y}}{dt} = \Lambda.\boldsymbol{y}$ if and only if there exists $F \in G = \exp(L)$ such that

$$d(F) = \mathrm{ad}_{x_0}(F) = Fu$$

Is d invertible ?

1. L is the completion of $\mathrm{Vect}\{x^{\boldsymbol{m}} \partial_{x_i}, |\boldsymbol{m}| \geqslant 2, 1 \leqslant i \leqslant \nu\}$ whose set $\{x^{\boldsymbol{m}} \partial_{x_i}, |\boldsymbol{m}| \geqslant 2, 1 \leqslant i \leqslant \nu\}$ is a linear basis.
2. For $\boldsymbol{m} = (m_1, \ldots, m_\nu)$,

$$\mathrm{ad}_{x_0}(x^{\boldsymbol{m}} \partial_{x_i}) = (m_1 \lambda_1 + \ldots + m_\nu \lambda_\nu - \lambda_i) x^{\boldsymbol{m}} \partial_{x_i} = (\langle \boldsymbol{m}, \lambda \rangle - \lambda_i) x^{\boldsymbol{m}} \partial_{x_i}$$

3. The derivation $d = \mathrm{ad}_{x_0}$ is diagonal in this basis and invertible iff the coefficients $\langle \boldsymbol{m}, \lambda \rangle - \lambda_i$ do not vanish : non-resonance of the vector field !
4. Note that, otherwise, the vector field is resonant and on L :

$$L = \mathrm{Ker}\, d \oplus d(L)$$

The non-invertible case.

In the resonant case, when $d = \mathrm{ad}_{x_0}$ is not invertible, the vector field $x_0 + u$ cannot be conjugated to the linear part x_0 but could be conjugated to another vector fields $x_0 + v$:

$$\mathrm{ad}_{x_0}(\varphi) + v\varphi = \varphi u$$

d-conjugacy.

Two elements in L are d-conjugate if there exists $\varphi \in G$ such that

$$d(\varphi) + v.\varphi = \varphi.u$$

φ d–conjugates u to v. This is an equivalence relation and we note $u \sim_d v$.

- → d is invertible on L : one class, the class of 0.
- → d non–invertible : non-trivial classes.

Conjugacy classes

Theorem 10. *Let d a graded derivation on L and $F=\prod_{n\geq 1} F_n$ a supplementary vector space of $d(L)$ in L. For any $u\in L$, there exists an element $u_F\in F$ which is a d-conjugate of u, that is $d(\varphi)+u_F\varphi=\varphi u$.*

Proof. We must find φ sucht that $\varphi^{-1}d(\varphi)+\varphi^{-1}u_F\varphi=u$
If $u=\sum_{n\geq 1} u_n$, $u_F=v=\sum_{n\geq 1} v_n\in F$ and $\varphi=\exp(\alpha)\in G$ $(\alpha=\sum_{n\geq 1}\alpha_n)$

$$\log_d(\exp(\alpha))+\exp(-\alpha)v\exp(\alpha)=\sum_{s\geq 0}\frac{(-1)^s}{(s+1)!}\mathrm{ad}_\alpha^s(d(\alpha))+\sum_{i\geq 0}\frac{(-1)^i}{i!}\mathrm{ad}_\alpha^i(v)=u$$

Thanks to the graduation, for $n\geqslant 1$,

$$d(\alpha_n)+P_{n-1}(\alpha,d(\alpha))+v_n+Q_{n-1}(\alpha,v)=u_n.$$

For $n=1$, this read $d(\alpha_1)=u_1-v_1$. If $v_1=p_F(u_1)$, $u_1-p_F(u_1)\in d(L)$, and we get a (not unique) solution to $d(\alpha_1)=u_1-v_1$. For $n=2$, $d(\alpha_2)-\frac{1}{2}[\alpha_1,d(\alpha_1)]+v_2-[\alpha_1,d(\alpha_1)]=u_2$: $v_2=p_F(u_2+\frac{1}{2}[\alpha_1,d(\alpha_1)]+[\alpha_1,d(\alpha_1)])$... end by recursion. □

Note that the element u_F is not unique. But, in the framework of resonant vector fields, one can choose $F = \ker d$ since

$$L = \ker d \oplus d(L).$$

We note p the projector on $\ker d$, parallel to $d(L)$.

Normalization.

When $L = \ker d \oplus d(L)$,For any $u \in \langle \mathrm{hL} \rangle$,

Theorem 11. *There exists* $v \in \ker d$ *such that*

$$u \sim_d v.$$

Moreover, v, $w \in \ker d$ *are* d*-conjugated to* u *if and only if there exists* $\varphi \in \exp(\ker d)$ *such that*

$$v\varphi = \varphi w$$

Such elements are called d-normal forms of u.

→ d-conjugacy classes identify to classical conjugacy classes of ker d by $\exp(\ker d)$.

→ If $u \sim_d 0$ (linearizable) there is a unique d–normal form for u : 0. $(\varphi^{-1} 0 \varphi = 0)$.

→ In general, there are several normal form but can we define some "universal" map $N\colon L \to \ker d$ such that $u \sim_d N(u)$ and N does not depend on u.

Normalization and renormalization.

Theorem 12. *Let d and δ two derivations on L such that:*

1. $L = \ker d \oplus d(L)$,
2. $\ker d$ *is stable by* δ,
3. *The restriction of* δ *from* $\ker d$ *to* $\ker d$ *is invertible.*

Consider $L_\varepsilon = L[[\varepsilon]][\varepsilon^{-1}]$ *the* $\mathcal{A} = \mathbb{C}[[\varepsilon]][\varepsilon^{-1}]$*–module over* L *(Lie algebra). For any* $u \in L \subset L_\varepsilon$*, the equation*

$$(d + \varepsilon\delta)\varphi = \varphi u \quad \text{(DimReg)}$$

has a unique solution $\varphi_\varepsilon \in G_\varepsilon = \exp(L_\varepsilon)$*.*
If $L_\varepsilon^+ = L[[\varepsilon]]$ $(G_\varepsilon^+ = \exp(L_\varepsilon^+))$ *and* $L_\varepsilon^- = \varepsilon^{-1}L[\varepsilon^{-1}]$ $(G_\varepsilon^- = \exp(L_\varepsilon^-))$*,* φ_ε *admits a unique (Birkhoff) factorisation in* $G_\varepsilon^- \times G_\varepsilon^+$*, that is* $\varphi_\varepsilon = \varphi_\varepsilon^- \varphi_\varepsilon^+$ *and* $\varphi_0^+ \in L$ *d–conjugates* u *to a d–normal form* $\beta \in \ker d$*. Moreover*

$$\delta\varphi_\varepsilon^- = \varepsilon^{-1}\varphi_\varepsilon^-\beta$$

Back to a toy model.

In the case $d = \mathrm{ad}_{x_1\partial_{x_1}}$ on $L = x_2^2\mathbb{C}[[x_1]]\partial_{x_2}$, one can choose $\delta = \mathrm{ad}_{x_1\partial_{x_1}+x_2\partial_{x_2}}$. In this case, $d+\varepsilon\delta = \mathrm{ad}_{(1+\varepsilon)x_1\partial_{x_1}+\varepsilon x_2\partial_{x_2}}$ and for a given element

$$u = a(x_1)x_2^2\partial_{x_2}, \qquad a(x_1) = \sum_{n\geq 0} a_n x_1^n$$

A solution of $(d+\varepsilon\delta)\varphi = \varphi u$ is given by $\varphi_\varepsilon = \exp(b(x_1)x_2^2\partial_{x_2})$ with

$$b(x_1) = \sum_{n\geq 0} \frac{a_n}{n(1+\varepsilon)+\varepsilon} x_1^n = \frac{a_0}{\varepsilon} + \sum_{n\geq 1} \frac{a_n}{n(1+\varepsilon)+\varepsilon} x_1^n$$

Thanks to the simplicity of the Lie algebra,

$$\varphi_\varepsilon^- = \exp\Big(\frac{a_0}{\varepsilon} x_2^2\partial_{x_2}\Big), \qquad \varphi_\varepsilon^+ = \exp\left(\sum_{n\geq 1} \frac{a_n}{n(1+\varepsilon)+\varepsilon} x_1^n x_2^2\partial_{x_2}\right)$$

so that a normal form of $u = a(x_1)x_2^2\partial_{x_2}$ is $a_0 x_2^2\partial_{x_2}$: The system $\begin{cases} \dot{x}_1 = x_1 \\ \dot{x}_2 = a(x_1)x_2^2 \end{cases}$ is formally conjugate to the normal form $\begin{cases} \dot{y}_1 = y_1 \\ \dot{y}_2 = a_0 y_2^2 \end{cases}$.

Proof of the theorem : regularization and invertibility.

Let d a graded derivation on L such that $L = \ker d \oplus d(L)$ and p, q the respective projections on the kernel of d and its image. The operators d, $d + \varepsilon\delta$, p and q extend to $L_\varepsilon = L[[\varepsilon]][\varepsilon^{-1}]$ and the restriction of d from $d(L)$ (or $d(L_\varepsilon)$) to itself is invertible and we note I this graded inverse. On the same way δ can be extended to L_ε, its restriction from $\ker d$ (or $\ker_\varepsilon d = \{u \in L_\varepsilon \,;\, d(u) = 0\}$) to itself is invertible and we note abusively δ^{-1} its inverse on $\ker d$ (or $\ker_\varepsilon d$).

Lemma 13. *The endomorphism $d + \varepsilon\delta$ is an invertible derivation on L_ε and its (graded) inverse is the linear map I_ε defined for $u \in L_\varepsilon$ by*

$$I_\varepsilon(u) = \varepsilon^{-1}\delta^{-1}(p(u)) + (\mathrm{Id} - \delta^{-1} \circ p \circ \delta) \circ I\left(\sum_{k \geq 0} (-1)^k \varepsilon^k (q \circ \delta \circ I)^{\circ^k}(q(u))\right)$$

Proof of the theorem : Birkhoff decomposition.

Since $d+\varepsilon\delta$ is invertible on L_ε, the equation $(d+\varepsilon\delta)\varphi = \varphi u$ has a solution $\varphi = \varphi_\varepsilon \in G_\varepsilon = \exp(L_\varepsilon)$ for any u in L_ε or L ($\varphi_\varepsilon = \exp_{d+\varepsilon\delta}(u)$). Thanks to the Birkhoff decomposition we have

$$\varphi_\varepsilon = \varphi_\varepsilon^- \varphi_\varepsilon^+, \quad \varphi_\varepsilon^\pm = \exp\left(\alpha_\varepsilon^\pm\right), \quad \alpha_\varepsilon^\pm \in L_\varepsilon^\pm$$

If $u \in L$, since $(d+\varepsilon\delta)\varphi_\varepsilon = \varphi_\varepsilon u$ we get

$$(d+\varepsilon\delta)\varphi_\varepsilon = (d+\varepsilon\delta)(\varphi_\varepsilon^- \varphi_\varepsilon^+) = \varphi_\varepsilon^-\big((d+\varepsilon\delta)\varphi_\varepsilon^+\big) + \big((d+\varepsilon\delta)\varphi_\varepsilon^-\big)\varphi_\varepsilon^+ = \varphi_\varepsilon^- \varphi_\varepsilon^+ u$$

and then

$$\big((d+\varepsilon\delta)\varphi_\varepsilon^+\big)(\varphi_\varepsilon^+)^{-1} + (\varphi_\varepsilon^-)^{-1}\big((d+\varepsilon\delta)\varphi_\varepsilon^-\big) = \varphi_\varepsilon^+ u (\varphi_\varepsilon^+)^{-1}$$

But $\beta = (\varphi_\varepsilon^-)^{-1}((d+\varepsilon\delta)\varphi_\varepsilon^-)$ is in $L[\varepsilon^{-1}]$ whereas the other terms are in $L[[\varepsilon]]$ thus these two parts of the identity do not depend on ε ! β is in L (not L_ε) and

$$((d+\varepsilon\delta)\varphi_\varepsilon^+ + \beta\varphi_\varepsilon^+ = \varphi_\varepsilon^+ u$$

For $\varepsilon = 0$ the element φ_0^+ conjugates u to β.

Proof of the theorem : normal form.

It remains to prove that $\beta = (\varphi_\varepsilon^-)^{-1}((d+\varepsilon\delta)\varphi_\varepsilon^-)$ is a normal form, that is $d(\beta) = 0$. Note that if $\varphi_\varepsilon^- = \exp(\alpha)$ with $d(\alpha) = 0$, then $\delta\varphi_\varepsilon^- = \varphi_\varepsilon^-(\varepsilon^{-1}\beta)$ with $d(\beta) = 0$:

$$\begin{array}{rcccc} \log_{d+\varepsilon\delta}(\varphi_\varepsilon^-) & = & \displaystyle\sum_{s\geq 0} \frac{(-1)^s}{(s+1)!}\mathrm{ad}_\alpha^s((d+\varepsilon\delta)(\alpha)) & = & \displaystyle\sum_{s\geq 0} \frac{(-1)^s}{(s+1)!}\mathrm{ad}_\alpha^s((\varepsilon\delta)(\alpha)) \\ & = & \log_{\varepsilon\delta}(\varphi_\varepsilon^-) & \in & \ker_\varepsilon d \\ & = & \beta & \in & L \end{array}$$

Lemma 14. *Let $\psi \in G_\varepsilon^-$. If*

$$\log_{d+\varepsilon\delta}(\psi) = \psi^{-1}.(d+\varepsilon\delta)(\psi) \in L \quad (\text{not } \; L_\varepsilon)$$

then $\psi = \exp(\alpha)$ with $d(\alpha) = 0$.

Using this lemma, together with the Birkhoff decomposition,

$$\beta = \log_{d+\varepsilon\delta}\varphi_\varepsilon^- \in L$$

is a normal form related to φ^- by

$$\delta\varphi^- = \varphi^-(\varepsilon^{-1}\beta).$$

Further developments.

Locality and Residues

The identity $\delta\varphi_\varepsilon^- = \varphi_\varepsilon^-(\varepsilon^{-1}\beta)$ has to be related to residues. If we write

$$\varphi_\varepsilon^- = 1 + \sum_{k\geq 1} \varepsilon^{-k}\varphi_k^-, \quad \varphi_k^- \in \mathcal{U}^k$$

then, the residue $\mathrm{Res}\,\varphi_\varepsilon^- = \varphi_1^-$ is such that

$$\delta\,\mathrm{Res}\varphi_\varepsilon^- = \beta$$

and it looks very close to the beta function in perturbative quantum field theory. This concept is related to the "locality" of counter terms (that is φ_ε^-). For a variable τ, consider the automorphism θ_τ:

$$\theta_\tau(x) = e^{\tau\varepsilon\delta}x$$

We say that $\varphi_\varepsilon \in G_\varepsilon$ is δ–local if $\varphi_\varepsilon^\tau = \theta_\tau(\varphi_\varepsilon) = (\varphi_\varepsilon^\tau)^-(\varphi_\varepsilon^\tau)^-$ is such that

$$\partial_\tau(\varphi_\varepsilon^\tau)^- = 0$$

This is indeed the case here ...

Another "renormalization" : the correction in dynamical systems.

In pQFT, the attempted (but ill-defined) group-like element φ is associated to a Lagrangian $\mathcal{L}$ and the renormalization procedure can be interpreted as an iterative process, based on the graduation of the considered Hopf algebra, that consists in modifying the Lagrangian:

$$\mathcal{L} \to \mathcal{L} - \mathcal{L}_1 \to \mathcal{L} - \mathcal{L}_1 - \mathcal{L}_2 \to \ldots \to \mathcal{L}^{\mathrm{ren}} = \mathcal{L} - \mathcal{L}^{\mathrm{ct}}$$

so that the renormalized group-like element φ^{ren} corresponds to the modified Lagrangian $\mathcal{L}^{\mathrm{ren}}$.
In the framework of dynamical systems:

Theorem 15. *Let d a derivation such that $L = \ker d \oplus d(L)$. For any $u \in L$, there exists a unique $u^c \in \ker d$ such that $u - u^c$ is in the d-conjugacy class of 0. u_c is called the correction of u. Moreover, if $u \in \ker d$, then $u^c = u$.*

As in the Lagrangian interpretation of renormalization in pQFT, the idea of the proof is to modify u, graded component by graded component, so that we can solve the equation:

$$d\varphi = \varphi(u - u_1^c - u_2^c - \ldots)$$

Prelie algebras ?

The same work shall be done in Prelie algebras. Roughly speaking, Prelie algebras (and Hopf algebras of trees) in dynamical systems allow to get information on analycity. Back to physics :

- Start with the coupling constant g:

$$S(\varphi)=\int_{\mathbb{R}^d}\left(-\frac{1}{2}\varphi(x).(-\Delta+m^2)(\varphi(x))\right)d^dx-g\int_{\mathbb{R}^d}\varphi^3(x)d^dx$$

- Compute an effective (more physical) coupling constant g^{eff}:

$$g^{\text{eff}}=\psi(g)=g+\sum_{n\geq 1}\psi_n g^{n+1},\ \psi_n \text{ polynomial in } I_d(\gamma)$$

- The renormalized effective coupling constant is given by:

$$\psi\to\psi_\varepsilon=\psi_\varepsilon^-\circ\psi_\varepsilon^+\to\psi_0^+(g)$$

- Theorem (FM, 2007) : If, for a given $N>0$, $\varepsilon^{-N}\psi_\varepsilon(\varepsilon^N g)$ is in $\mathbb{C}\{\varepsilon,\ g\}$, then $\psi_0^+(g)$ is analytic.

Dynamical systems and multizetas.

Consider a diffeomorphism (z at infinity, x near 0) : $F(z,x) = (z+1, x + \frac{1}{z^{d+1}})$ that looks like a perturbation of $T(z,x) = (z+1, x)$.
Is there a change of coordinates (conjugacy) $(z,x) = \Phi(z,y) = (z, y + u(z))$ such that $F = \Phi \circ T \circ \Phi^{-1}$ or $F \circ \Phi = \Phi \circ T$:

$$F(z, y+u(z)) = (z+1, x+u(z)+\frac{1}{z^{d+1}}) = (z+1, y+u(z+1)) = \Phi(z+1, y)$$

Thus $u(z+1) - u(z) = \frac{1}{z^{d+1}}$.

$$u(z) = -\sum_{n \geqslant 0} \frac{1}{(z+n)^{1+d}} \qquad ???$$

- Problem for $d=0$: renormalisation ? $d = \varepsilon$ or

$$u(z) = -\frac{1}{z} - \left(\sum_{n \geqslant 1} \frac{1}{(z+n)} - \frac{1}{n} \right)$$

- For more complicated diffeomorphisms $F(z,x) = (z+1, a(x, z^{-1}))$, we get Hurwitz multizetas ....

Discrete dynamical systems

Near the origin, consider a diffeomorphism

$$F(\boldsymbol{x}) = F(x_1, \ldots, x_\nu) = (\ell_1 x_1 + h.o.t, \ldots, \ell_\nu x_\nu + h.o.t)$$

In order to study the dynamics of F°^n}, is there a diffeomorphim $\boldsymbol{x} = \Psi(\boldsymbol{y})$ such that

$$\Psi^{-1} \circ F \circ \Psi(\boldsymbol{y}) = (\ell_1 y_1, \ldots, \ell_\nu y_\nu)$$

By analogy with derivations $d = \mathrm{ad}_{x_0}$ and the equation $d(\varphi) = \varphi u$, one should study in a Lie group the equation :

$$\theta(\psi) = \psi\varphi$$

where $\varphi, \psi \in G$ and $\theta([x, y]) = [\theta(x), \theta(y)]$ $(\theta(1) = 1)$: if $\psi = 1 + u$, $\varphi = 1 + v$ then

$$(\theta - \mathrm{Id})u = v + uv$$

Condition on the invertibility of $\theta - \mathrm{Id}$ ...