Multiple zeta values and their q-analogues

Joint work with K. Ebrahimi-Fard and J. Castillo-Medina

D. Manchon

Potsdam, November 7-9th 2013

Multiple zeta values are given by the following iterated series:

$$\zeta(n_1,\ldots,n_k)=\sum_{m_1>\cdots>m_k>0}\frac{1}{m_1^{n_1}\cdots m_k^{n_k}}. \qquad (1)$$

- The n_j's are positive integers.
- The series converges provided $n_1\geq 2$.
- The integer k is the **depth**, the sum $w:=n_1+\cdots+n_k$ is the **weight**.

Quasi-shuffle relations

The product of two MZVs is a linear combination of MZVs!
For example:

$$\begin{aligned}\zeta(n_1)\zeta(n_2) &= \sum_{m_1>m_2>0} \frac{1}{m_1^{n_1} m_2^{n_2}} + \sum_{m_2>m_1>0} \frac{1}{m_1^{n_1} m_2^{n_2}} + \sum_{m_1=m_2>0} \frac{1}{m_1^{n_1} m_2^{n_2}} \\ &= \zeta(n_1, n_2) + \zeta(n_2, n_1) + \zeta(n_1 + n_2).\end{aligned}$$

The most general **quasi-shuffle relation** displays as follows:

$$\zeta(n_1,\ldots,n_p)\zeta(n_{p+1},\ldots,n_{p+q}) = \sum_{r\geq 0}\quad \sum_{\sigma\in \mathrm{qsh}(p,q;r)} \zeta(n_1^\sigma,\ldots,n_{p+q-r}^\sigma).$$

- Here $\mathrm{qsh}(p,q;r)$ stands for (p,q)**-quasi-shuffles of type** r. They are surjections

  $$\sigma : \{1,\ldots,p+q\} \longrightarrow\!\!\!\!\!\rightarrow \{1,\ldots,p+q-r\}$$

  subject to $\sigma_1 < \cdots < \sigma_p$ and $\sigma_{p+1} < \cdots < \sigma_{p+q}$.
- n_j^σ stands for the **sum** of the n_r's for $\sigma(r) = j$.
- The sum above contains only one or two terms.

Integral representation and shuffle relations

MZVs have an iterated integral representation:

$$\zeta(n_1,\ldots,n_k)=\int_{0\le t_w\le\cdots\le t_1\le 1}\frac{dt_1}{t_1}\cdots\frac{dt_{n_1-1}}{t_{n_1-1}}\frac{dt_{n_1}}{1-t_{n_1}}\cdots\frac{dt_{n_1+\cdots+n_{k-1}}}{t_{n_1+\cdots+n_{k-1}}}\cdots\frac{dt_{w-1}}{t_{w-1}}\frac{dt_w}{1-t_w}$$

As a consequence, there is a second way to express the product of two MZVs as a linear combination of MZVs: the **shuffle relations**.

Example:

$$\begin{aligned}\zeta(2)\zeta(2) &= \int_{\substack{0\le t_2\le t_1\le 1\\ 0\le t_4\le t_3\le 1}}\frac{dt_1}{t_1}\frac{dt_2}{1-t_2}\frac{dt_3}{t_3}\frac{dt_4}{1-t_4}\\ &= 4\zeta(3,1)+2\zeta(2,2).\end{aligned}$$

Regularization relations

A third group of relations can be deduced from the preceding ones: the **regularization relations**. The simplest one is:

$$\zeta(2,1) = \zeta(3).$$

These three groups of relations constitute the so-called **double shuffle relations**.

It is conjectured that no other relations occur among multiple zeta values. Only tiny steps have been done in that direction.

- Introduce two alphabets $X := \{x_0, x_1\}$, $Y := \{y_1, y_2, y_3, \ldots\}$.
- X^* (resp. Y^*) is the set of words with letters in X (resp. Y).
- $\mathbb{Q}\langle X\rangle$ (resp. $\mathbb{Q}\langle Y\rangle$) linear span of X^* (resp. Y^*) on $\mathbb{Q}$.
- **Shuffle product** on $\mathbb{Q}\langle X\rangle$:

$$v_1 \cdots v_p ⧢ v_{p+1} \cdots v_{p+q} := \sum_{\sigma \in \mathrm{sh}(p,q)} v_{\sigma_1^{-1}} \cdots v_{\sigma_{p+q}^{-1}}.$$

- **Quasi-shuffle product** on $\mathbb{Q}\langle Y\rangle$:

$$u_1 \cdots u_p \,\overline{⧢}\, u_{p+1} \cdots u_{p+q} := \sum_{r \geq 0} \sum_{\sigma \in \mathrm{qsh}(p,q;r)} u_1^\sigma \cdots u_{p+q-r}^\sigma,$$

where u_j^σ is the **internal product** of the u_r's with $\sigma(r) = j$. The internal product is given by $y_i \diamond y_j = y_{i+j}$.

- Notation: $X^*_{\text{conv}} := x_0 X^* x_1,$ $Y^*_{\text{conv}} := Y^* \setminus y_1 Y^*.$
- change of coding (**swap**):

$$\begin{array}{rcl} \mathfrak{s} : Y^* & \longrightarrow & X^* \\ y_{n_1} \cdots y_{n_k} & \longmapsto & x_0^{n_1 - 1} x_1 \cdots x_0^{n_k - 1} x_1 . \end{array}$$

Clearly, $\mathfrak{s}(Y^*) = X^* x_1$ and $\mathfrak{s}(Y^*_{\text{conv}}) = X^*_{\text{conv}}$.

- For any word $y_{n_1} \cdots y_{n_k}$ in Y^*_{conv} we set:

$$\zeta_{⧢\!\!\!\!+}(y_{n_1} \cdots y_{n_k}) := \zeta(n_1, \cdots n_k) =: \zeta_{⧢}(x_0^{n_1 - 1} x_1 \cdots x_0^{n_k - 1} x_1).$$

- As a consequence we have on X^*_{conv}:

$$\zeta_{⧢\!\!\!\!+} = \zeta_{⧢} \circ \mathfrak{s}.$$

- Extend $\zeta_{⧢}$ and $\zeta_{⧢\!\!\!\!+}$ linearly.

- By considering the ill-defined quantity $\zeta(1)$ as an indeterminate θ, it is possible to extend both $\zeta_{ш}$ and $\zeta_{⧢}$ to all X^*x_1 and Y^* respectively, in a unique way, such that:
  - $\zeta_{ш}(v ш v') = \zeta_{ш}(v)\zeta_{ш}(v')$ for any $v, v' \in X^*x_1$.
  - $\zeta_{⧢}(u ⧢ u') = \zeta_{⧢}(u)\zeta_{⧢}(u')$ for any $u, u' \in Y^*$.

- The relation $\zeta_{⧢} = \zeta_{ш} \circ \mathfrak{s}$ is no longer true on X^*x_1, but there is an infinite order differential operator $\rho : \mathbb{R}[\theta] \longrightarrow \mathbb{R}[\theta]$ with constant coefficients such that:

$$\zeta_{ш} \circ \mathfrak{s} = \rho \circ \zeta_{⧢}.$$

  (D. Zagier, L. Boutet de Monvel). **Regularization relations** come from there.

- If desired, extend $\zeta_{ш}$ to all X^*. A good choice is $\zeta_{ш}(x_0) = \theta$.

Multiple polylogarithms

For any $t \in [0,1]$,

$$\mathrm{Li}_{n_1,\ldots,n_k}(t) := \int\limits_{0\le t_w\le\cdots\le t_1\le t} \frac{dt_1}{t_1} \cdots \frac{dt_{n_1-1}}{t_{n_1-1}} \frac{dt_{n_1}}{1-t_{n_1}} \cdots \frac{dt_{n_1+\cdots+n_{k-1}}}{t_{n_1+\cdots+n_{k-1}}} \cdots \frac{dt_{w-1}}{t_{w-1}} \frac{dt_w}{1-t_w}$$

$$= \sum_{m_1>\cdots>m_k>0} \frac{t^{m_1}}{m_1^{n_1}\cdots m_k^{n_k}}.$$

$$x(t) := \tfrac{1}{t}, \qquad\qquad y(t) := \tfrac{1}{1-t}.$$

Three operators on the space of continuous maps $f : [0,1] \to \mathbb{R}$:
$X[f](t) := x(t)f(t), \quad Y[f](t) := y(t)f(t), \quad R[f](t) := \int_0^t f(u)\,du.$

$\Rightarrow$ **Concise expression** of the multiple polylogarithm:

$$\mathrm{Li}_{n_1,\ldots,n_k} = (R\circ X)^{n_1-1}\circ(R\circ Y)\circ\cdots\circ(R\circ X)^{n_k-1}\circ(R\circ Y)[\mathbf{1}].$$

R is a **weight zero Rota-Baxter operator**:

$$R[f]R[g] = R\big[R[f]g + fR[g]\big].$$

We have of course for any positive integers $n_1, \ldots n_k$ with $n_1 \geq 2$:

$$\mathrm{Li}_{n_1,\ldots,n_k}(1) = \zeta(n_1, \ldots, n_k).$$

Historical remarks

- Double zeta values were already known by **L. Euler**, as well as all the relations above relating double and simple ones.
- MZVs in full generality seem to appear for the first time in the work of **J. Ecalle** (*Les fonctions résurgentes*, Univ. Orsay, 1981).
- Growing interest since the works of **D. Zagier** and **M. Hoffman** (early 90's).
- Integral representation attributed to **M. Kontsevich** (D. Zagier, 1994), starting point of the modern approach (periods of mixed Tate motives...).
- Recent breakthrough by **F. Brown** (2012):
  Any MZV is a linear combination, with rational coefficients, of MZVs with arguments equal to 2 or 3.

The **Jackson integral** is defined by:

$$J[f](t) = \int_0^t f(u)\, d_q u = \sum_{n \geq 0} (q^n t - q^{n+1} t) f(q^n t).$$

- Here q is a parameter in $]0,1[$.
- When $q \nearrow 1$ the Riemann sum above converges to the ordinary integral.
- q can also be considered as an indeterminate: The Jackson integral operator J is then a $\mathbb{Q}[[q]]$-linear endomorphism of

$$\mathcal{A} := t\mathbb{Q}[[t,q]].$$

A weight -1 Rota-Baxter operator

The $\mathbb{Q}[[q]]$-linear operator $P_q : \mathcal{A} \longrightarrow \mathcal{A}$ defined by:

$$P_q[f](t) := \sum_{n \geq 0} f(q^n t) = f(t) + f(qt) + f(q^2 t) + f(q^3 t) + \cdots$$

satisfies the **weight -1 Rota-Baxter identity:**

$$P_q[f]P_q[g] = P_q\big[P_q[f]g + fP_q[g] - fg\big].$$

Operator P_q is **invertible** with inverse:

$$P_q^{-1}[f](t) = D_q[f](t) = f(t) - f(qt).$$

The q-difference operator D_q satisfies a modified Leibniz rule:

$$D_q[fg] = D_q[f]g + fD_q[g] - D_q[f]D_q[g].$$

We end up with **three identities**:

$$\begin{aligned} P_q[f]P_q[g] &= P_q\big[P_q[f]g + fP_q[g] - fg\big], \\ D_q[f]D_q[g] &= D_q[f]g + fD_q[g] - D_q[fg], \\ D_q[f]P_q[g] &= D_q\big[fP_q[g]\big] + D_q[f]g - fg. \end{aligned}$$

Multiple q-polylogarithms

- Introduce the functions:

$$x(t) := \frac{1}{t}, \qquad y(t) := \frac{1}{1-t}, \qquad \overline{y}(t) := \frac{t}{1-t}.$$

Note that $\overline{y}$ is an element of $\mathcal{A}$.

- Introduce $X, Y, \overline{Y}$, multiplication operators by $x, y, \overline{y}$ resp.
- Recall:

$$\mathrm{Li}_{n_1,\ldots,n_k} = (R \circ X)^{n_1-1} \circ (R \circ Y) \circ \cdots \circ (R \circ X)^{n_k-1} \circ (R \circ Y)[\mathbf{1}].$$

- **Analogously**:

$$\mathrm{Li}^{q}_{n_1,\ldots,n_k} := (J \circ X)^{n_1-1} \circ (J \circ Y) \circ \cdots \circ (J \circ X)^{n_k-1} \circ (J \circ Y)[\mathbf{1}].$$

Ohno-Okuda-Zudilin q-multiple zeta values

- Recall:

$$\zeta(n_1,\ldots,n_k)=\mathrm{Li}_{n_1,\ldots,n_k}(1).$$

- By analogy define:

$$\mathfrak{z}_q(n_1,\ldots,n_k):=\mathrm{Li}^{q}_{n_1,\ldots,n_k}(q).$$

- Some straightforward computation shows:

$$\mathfrak{z}_q(n_1,\ldots,n_k)=\sum_{m_1>\cdots>m_k}\frac{q^{m_1}}{[m_1]_q^{n_1}\cdots[m_k]_q^{n_k}},$$

with usual q-numbers:

$$[m]_q=\frac{1-q^m}{1-q}=1+q+\cdots+q^{m-1}.$$

- For any positive integers $n_1, \ldots n_k$ with $n_1 \geq 2$, the q-MZV $\mathfrak{z}_q(n_1, \ldots, n_k)$ makes sense for any complex q with $|q| \leq 1$, and we have:
$$\lim_{q \to 1} \mathfrak{z}_q(n_1, \ldots, n_k) = \zeta(n_1, \ldots, n_k).$$
- An alternative description in terms of the operator P_q will be very convenient:

$$\begin{aligned}
\bar{\mathfrak{z}}_q(n_1, \ldots, n_k) &:= (1-q)^{-w} \mathfrak{z}_q(n_1, \ldots, n_k) \\
&= \sum_{m_1 > \cdots > m_k > 0} \frac{q^{m_1}}{(1-q^{m_1})^{n_1} \cdots (1-q^{m_k})^{n_k}} \\
&= P_q^{n_1} \circ \overline{Y} \circ \cdots \circ P_q^{n_1} \circ \overline{Y}[\mathbf{1}](t)\big|_{t=q}.
\end{aligned}$$

Extension to arguments of any sign

- The iterated sum defining $\bar{\mathfrak{z}}_q(n_1, \ldots, n_k)$ makes perfect sense in $\mathbb{Q}[[q]]$ for any $n_1, \ldots, n_k \in \mathbb{Z}$.
- moreover it also makes sense when specializing q to a complex number of modulus < 1:

$$|\bar{\mathfrak{z}}_q(n_1, \ldots, n_k)| \leq |q|^k (1 - |q|)^{-w'-k},$$

with $w' := \sum_{i=1}^k \sup(0, n_i)$.

- **For any $n_1, \ldots, n_k \in \mathbb{Z}$ we still have (with $P_q^{-1} = D_q$):**

$$\bar{\mathfrak{z}}_q(n_1, \ldots, n_k) = P_q^{n_1} \circ \overline{Y} \circ \cdots \circ P_q^{n_1} \circ \overline{Y}[\mathbf{1}](t)\big|_{t=q}.$$

Examples

$$\begin{aligned}
\bar{\mathfrak{z}}_q(0) &= \frac{q}{1-q}, \\
\bar{\mathfrak{z}}_q(\underbrace{0,\ldots,0}_{k}) &= \left(\frac{q}{1-q}\right)^k, \\
\bar{\mathfrak{z}}_q(-1) &= \sum_{m>0} q^m(1-q^m) = \frac{q}{1-q} - \frac{q^2}{1-q^2}.
\end{aligned}$$

q-shuffle relations

- Let $\widetilde{X}$ be the alphabet $\{d, y, p\}$.
- Let W be the set of words on the alphabet $\widetilde{X}$, ending with y and subject to
$$dp = pd = \mathbf{1},$$
where $\mathbf{1}$ is the empty word.
- Any nonempty word in W writes uniquely $v = p^{n_1}y \cdots p^{n_k}y$, with $n_1, \ldots, n_k \in \mathbb{Z}$.
- Now define:
$$\bar{\mathfrak{z}}_q^{\sqcup\sqcup}(p^{n_1}y \cdots p^{n_k}y) := \bar{\mathfrak{z}}_q(n_1, \ldots, n_k)$$
and extend linearly.

- q-shuffle product recursively given (w.r.t. length of words) by $\mathbf{1} \shuffle v = v \shuffle \mathbf{1} = v$ and:

$$\begin{aligned}
(yv) \shuffle u = v \shuffle (yu) &= y(v \shuffle u), \\
dv \shuffle du &= v \shuffle du + dv \shuffle u - d(v \shuffle u), \\
pv \shuffle pu &= p(v \shuffle pu) + p(pv \shuffle u) - p(v \shuffle u), \\
dv \shuffle pu = pu \shuffle dv &= d(v \shuffle pu) + dv \shuffle u - v \shuffle u.
\end{aligned}$$

  for any $u, v \in W$.
- The product $\shuffle$ is **commutative** and **associative**.
- The q-shuffle relations write:

$$\bar{\mathfrak{z}}_q^{\shuffle}(u)\bar{\mathfrak{z}}_q^{\shuffle}(u) = \bar{\mathfrak{z}}_q^{\shuffle}(u \shuffle v).$$

q-quasi-shuffle relations

- $\widetilde{Y} =$ alphabet $\{z_n,\, n \in \mathbb{Z}\}$, with internal product $z_i \diamond z_j = z_{i+j}$.
- Let $\widetilde{Y}^*$ be set of words with letters in $\widetilde{Y}$.
- Let $*$ be the ordinary quasi-shuffle product on $\mathbb{Q}\langle \widetilde{Y} \rangle$.
- Let T be the shift operator defined for any word u by:

$$T(z_n u) := z_{n-1} u.$$

- The q-quasi-shuffle product $\uplus$ is (uniquely) defined by:

$$T(u \uplus v) = Tu * Tv.$$

- Define $\bar{\mathfrak{z}}_q^{\,\uplus}(z_{n_1}\cdots z_{n_k}) := \bar{\mathfrak{z}}_q(n_1,\ldots,n_k)$ and extend linearly.
- the q-quasi-shuffle relations write:

$$\bar{\mathfrak{z}}_q^{\,\uplus}(u)\bar{\mathfrak{z}}_q^{\,\uplus}(v) = \bar{\mathfrak{z}}_q^{\,\uplus}(u \uplus v)$$

  for any words $u, v \in \widetilde{Y}^*$.
- Example of q-quasi-shuffle relation: for any $a, b \in \mathbb{Z}$,

$$\begin{aligned}\bar{\mathfrak{z}}_q(a)\bar{\mathfrak{z}}_q(b) &= \bar{\mathfrak{z}}_q(a,b)+\bar{\mathfrak{z}}_q(b,a)+\bar{\mathfrak{z}}_q(a+b)\\ &\quad -\bar{\mathfrak{z}}_q(a,b-1)-\bar{\mathfrak{z}}_q(b,a-1)-\bar{\mathfrak{z}}_q(a+b-1).\end{aligned}$$

- Note that the weight is **not** conserved, contrarily to the classical case.

- In terms on "non-modified" q-MZVs, the previous example becomes:

$$\begin{aligned}\mathfrak{z}_q(a)\mathfrak{z}_q(b) &= \mathfrak{z}_q(a,b)+\mathfrak{z}_q(b,a)+\mathfrak{z}_q(a+b)\\ &\quad -(1-q)\big[\mathfrak{z}_q(a,b-1)-\mathfrak{z}_q(b,a-1)-\mathfrak{z}_q(a+b-1)\big].\end{aligned}$$

- In the limit $q \to 1$, the "weight drop term" disappears, and we recover the classical quasi-shuffle relation.

Important remark

There are no regularization relations in this picture. The swap

$$\mathfrak{r} : \widetilde{Y}^* \to W$$

is defined by:

$$\mathfrak{r}(z_{n_1} \cdots z_{n_k}) := p^{n_1 - 1} y \cdots p^{n_k - 1} y,$$

and the change of coding writes itself:

$$\overline{\mathfrak{z}}_q^{\,\uplus} = \overline{\mathfrak{z}}_q^{\,\sqcup\!\sqcup} \circ \mathfrak{r}$$

in full generality.

Summing up, the double q-shuffle relations write themselves as follows:

for any $u, v \in \widetilde{Y}^*$ and for any $u', v' \in W$,

$$\begin{aligned} \bar{\mathfrak{z}}_q^{\,\uplus}(u)\bar{\mathfrak{z}}_q^{\,\uplus}(v) &= \bar{\mathfrak{z}}_q^{\,\uplus}(u \uplus v), \\ \bar{\mathfrak{z}}_q^{\,\sqcup\!\sqcup}(u')\bar{\mathfrak{z}}_q^{\,\sqcup\!\sqcup}(v') &= \bar{\mathfrak{z}}_q^{\,\uplus}(u' \sqcup\!\sqcup v'), \end{aligned}$$

and we also have:

$$\bar{\mathfrak{z}}_q^{\,\uplus} = \bar{\mathfrak{z}}_q^{\,\sqcup\!\sqcup} \circ \mathfrak{r}.$$

References

- Y. Ohno, J.-I. Okuda, W. Zudilin,
  Cyclic q-MZV sum,
  J. Number Theory **132** (2012), 144–155.
- Y. Takeyama,
  The algebra of a q-analogues of multiple harmonic series,
  `arXiv:1306.6164` (2013).
- J. Castillo-Medina, K. Ebrahimi-Fard, D. Manchon,
  Unfolding the double shuffle structure of q-MZVs,
  `arXiv:1310.1330` (2013).

Thank you for your attention!