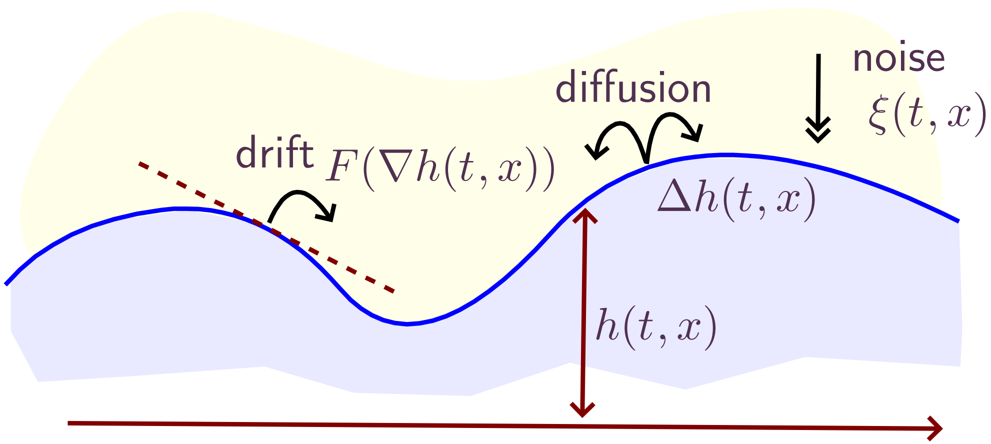
Kardar, Parisi and Zhang ('80) introduced an equation for the large scale dynamics of a growing interface. It takes into account three effects: drift, diffusion, noise :

$$\partial_t h(t,x) = \Delta h(t,x) + F(\partial_x h(t,x)) + \xi(t,x), \qquad x \in \mathbb{T}, t \geqslant 0$$



$$\langle \xi(t,x)\xi(s,y)\rangle = \delta(t-s)\,\delta(x-y)$$

Expansion aroung a flat interface :

$$F(\partial_x h(t,x)) = c_0 + c_1\,\partial_x h(t,x) + c_2(\partial_x h(t,x))^2 + \cdots$$

$$\partial_t h(t,x) = \Delta h(t,x) + c_0 + c_1 \partial_x h(t,x) + c_2(\partial_x h(t,x))^2 + \cdots + \xi(t,x), \qquad x \in \mathbb{T}, t \geqslant 0$$

The c_0 and c_1 contributions can be eliminated by the Galileian transformation

$$h(t,x) \to h(t, x - c_1 t) + c_0 t$$

and fix $c_2 = 1$ by rescaling. Finally drop all the higher order contributions to get the **KPZ** equation

$$\partial_t h(t,x) = \Delta h(t,x) + (\partial_x h(t,x))^2 + \xi(t,x), \qquad x \in \mathbb{T}, t \geqslant 0$$

Setting $u = \partial_x h$ we get the **stochastic Burgers equation**

$$\partial_t u(t,x) = \Delta u(t,x) + \partial_x (u(t,x))^2 + \partial_x \xi(t,x), \qquad x \in \mathbb{T}, t \geqslant 0$$

which is equivalent to KPZ.

▷ **Invariant measure:** Formally the SBE leaves invariant the space white noise: if u_0 has a Gaussian distribution with covariance $\mathbb{E}[u_0(x)u_0(y)] = \delta(x-y)$ then for all $t \geqslant 0$ the random function $u(t,\cdot)$ has a Gaussian law with the same covariance.

▷ **First order approximation:** Let $X(t,x)$ be the solution of the linear equation

$$\partial_t X(t,x) = \Delta X(t,x) + \partial_x \xi(t,x), \qquad x \in \mathbb{T}, t \geqslant 0$$

given by

$$X(t,x) = \int_{-\infty}^{t} \mathrm{d}r \int_{\mathbb{T}} \mathrm{d}y \, \partial_x p_{t-r}(x-y) \xi(r,y)$$

Then X is a stationary Gaussian process with covariance

$$\mathbb{E}[X(t,x)\, X(s,y)] = p_{|t-s|}(x-y).$$

A computation gives that, almost surely $X(t,\cdot) \in \mathscr{C}^{\gamma}$ for any $\gamma < -1/2$ and any $t \in \mathbb{R}$. Note also that for any $t \in \mathbb{R}$ $X(t,\cdot)$ has the law of the white noise over $\mathbb{T}$.

Higher order approximations

- ▷ Let $u = X + u_1$ and $L = \partial_t - \Delta$ then $Lu = \partial_x u^2 + \partial_x \xi$, $LX = \xi$ and

$$Lu_1 = \partial_x (u_1 + X)^2 = \underbrace{\partial_x X^2}_{-2-} + 2\partial_x (u_1 X) + \partial_x u_1^2$$

- ▷ Let $X^{\vee}$ be the solution to

$$LX^{\vee} = \partial_x X^2 \qquad \Rightarrow \qquad X^{\vee} \in \mathscr{C}^{0-}$$

and decompose further $u_1 = X^{\vee} + u_2$. Then

$$Lu_2 = \underbrace{2\partial_x (X^{\vee} X)}_{-3/2-} + 2\partial_x (u_2 X) + \underbrace{\partial_x (X^{\vee} X^{\vee})}_{-1-} + 2\partial_x (u_2 X^{\vee}) + \partial_x (u_2)^2$$

- ▷ Define $LX^{\curlyvee} = 2\partial_x (X^{\vee} X)$ and $u_2 = X^{\curlyvee} + u_3$ then $X^{\curlyvee} \in \mathscr{C}^{1/2-}$

$$Lu_3 = \underbrace{2\partial_x (u_3 X)}_{-3/2-} + \underbrace{2\partial_x (X^{\curlyvee} X)}_{-1-} + \underbrace{\partial_x (X^{\vee} X^{\vee})}_{-1-} + 2\partial_x (u_2 X^{\vee}) + \partial_x (u_2)^2$$

▷ **Binary trees.** The expansion generates a certain number of explicit terms, obtained via various combinations of X and of a bilinear map B given by

$$L B(f, g) = \partial_x (f g)$$

These terms can be described in terms of binary trees. A binary tree $\tau \in \mathcal{T}$ is either the root $\bullet$ or the combination of two smaller binary trees $\tau = (\tau_1 \tau_2)$. The natural grading $d: \mathcal{T} \to \mathbb{N}$ is given by $d(\bullet) = 0$ and $d((\tau_1 \tau_2)) = 1 + d(\tau_1) + d(\tau_2)$.

Define recursively a map $X: \mathcal{T} \to C(\mathbb{R}_+; \mathcal{S}'(\mathbb{T}))$ by

$$X^{\bullet} = X, \qquad X^{(\tau_1 \tau_2)} = B(X^{\tau_1}, X^{\tau_2})$$

giving

$$X^{\vee} = B(X, X), \quad X^{(\vee\bullet)} = B(X, X^{\vee}), \quad X^{(\bullet(\vee\bullet))} = B(X, X^{(\vee\bullet)}), \quad X^{(\vee\vee)} = B(X^{\vee}, X^{\vee})$$

and so on, where

Formal expansion

▷ We observe that formally

$$u = \sum_{\tau \in \mathcal{T}} c(\tau) X^{\tau}$$

where $c(\tau)$ is a combinatorial factor counting the number of planar trees which are isomorphic (as graphs) to τ. For example $c(\bullet) = 1$, $c($$) = 1$, $c($$) = 2$, $c\Big($$\Big) = 4$, $c($$) = 1$ and in general $c(\tau) = \sum_{\tau_1, \tau_2 \in \mathcal{T}} \mathbb{I}_{(\tau_1 \tau_2) = \tau} c(\tau_1) c(\tau_2)$.

▷ We can also write an equation for the truncated series. Setting

$$u = \sum_{\tau \in \mathcal{T}, d(\tau) < n} c(\tau) X^{\tau} + U$$

we have that the equation satisfied by U is obtained from the fixed point equation

$$u = X + B(u, u)$$

and reads

$$U = \sum_{\substack{\tau_1, \tau_2 : d(\tau_1) < n, d(\tau_2) < n \\ d((\tau_1 \tau_2)) \geqslant n}} c(\tau_1) c(\tau_2) B(X^{\tau_1}, X^{\tau_2}) + \sum_{\tau : d(\tau) < n} c(\tau) B(X^{\tau_1}, U) + B(U, U).$$

▷ The process X has the integral representation

$$X(t,x) = \int_{\mathbb{R}\times E} e^{i\xi x} H_{t-s}(\xi) W(\mathrm{d}\eta)$$

where $\eta = (s,\xi) \in \mathbb{R}\times E$, $E = \mathbb{Z}\backslash\{0\}$, $h_t(\xi) = e^{-\xi^2 t}\mathbb{I}_{t\geqslant 0}$, $H(t,\xi) = i\xi h_t(\xi)$ and $W(\mathrm{d}\eta)$ is the complex Gaussian process on $\mathbb{R}\times E$ defined by the covariance

$$\mathbb{E}\left(\int_{\mathbb{R}\times E} f(\eta)W(\mathrm{d}\eta)\int_{\mathbb{R}\times E} g(\eta')W(\mathrm{d}\eta')\right) = \int_{\mathbb{R}\times E} g(\eta_1)f(\eta_{-1})\mathrm{d}\eta_1$$

where $\eta_a = (s_a, \xi_a)$, $s_{-a} = s_a$, $\xi_{-a} = -\xi_a$, $\mathrm{d}\eta_a = \mathrm{d}s_a\mathrm{d}\xi_a$ is the product of the Lebesgue measure $\mathrm{d}s_a$ on $\mathbb{R}$ and of the counting measure $\mathrm{d}\xi_a$ on $E = \mathbb{Z}\backslash\{0\}$. The function f, g are complex functions in $L^2(\mathbb{R}\times E)$ satisfying $f(\eta_{-1}) = f(\eta_1)^*$.

▷ **Example.** The covariance of X can be computed as

$$\mathbb{E}[X(t,x)X(s,y)] = \int_E \mathrm{d}\xi_1 e^{i\xi_1(x-y)}\int_{\mathbb{R}} H_{t-s_1}(\xi_1)H_{s-s_2}(-\xi_1)\mathrm{d}s_1$$

$$= \int_E e^{i\xi_1(x-y)}\frac{e^{-\xi_1^2|t-s|}}{2}\mathrm{d}\xi_1 = \frac{1}{2}p_{|t-s|}(x-y)$$

▷ Recall that $X^\bullet = X$ and $X^{(\tau_1\tau_2)} = B(X^{\tau_1}, X^{\tau_2})$. Then

$$X^\tau(t,x) = \int_{(\mathbb{R}\times E)^n} G^\tau(t,x,\eta_\tau) \prod_{i=1}^n W(\mathrm{d}\eta_i)$$

where $n = d(\tau)+1$, $\eta_\tau = \eta_{1\ldots n} = (\eta_1, \ldots, \eta_n) \in (\mathbb{R}\times E)^n$ and $\mathrm{d}\eta_\tau = \mathrm{d}\eta_{1\ldots n} = \mathrm{d}\eta_1 \cdots \mathrm{d}\eta_n$. Here we mean that each of the X^τ is a polynomial in the Gaussian variables $W(\mathrm{d}\eta_i)$.

▷ The kernels G^τ are defined recursively by

$$G^\bullet(t,x,\eta) = e^{i\xi x} H_{t-s}(\xi)$$

$$G^{(\tau_1\tau_2)}(t,x,\eta_{(\tau_1\tau_2)}) = B(G^{\tau_1}(\cdot,\cdot,\eta_{\tau_1}), G^{\tau_2}(\cdot,\cdot,\eta_{\tau_2}))(t,x)$$

$$= \int_{-\infty}^t \mathrm{d}\sigma\, \partial_x P_{t-s}(G^{\tau_1}(\sigma,\cdot,\eta_{\tau_1}), G^{\tau_2}(\sigma,\cdot,\eta_{\tau_2}))(x)$$

Kernels (cont.)

▷ In the first few cases this gives

$$G^{\vee}(t,x,\eta_{12}) = \int_0^t \mathrm{d}\sigma\, \partial_x P_{t-\sigma}(G^{\bullet}(\sigma,\cdot,\eta_1), G^{\bullet}(\sigma,\cdot,\eta_2))(x)$$

$$= e^{i\xi_{[12]}x} \int_0^t H_{t-\sigma}(\xi_{[12]}) H_{\sigma-s_1}(\xi_1) H_{\sigma-s_2}(\xi_2) \mathrm{d}\sigma$$

where we set $\xi_{[1\cdots n]} = \xi_1 + \cdots + \xi_n$.

$$G^{\curlyvee}(t,x,\eta_{123}) = \int_0^t \mathrm{d}\sigma\, \partial_x P_{t-\sigma}(G^{\vee}(\sigma,\cdot,\eta_{12}), G^{\bullet}(\sigma,\cdot,\eta_3))(x)$$

$$= e^{i\xi_{[123]}x} \int_0^t \mathrm{d}\sigma \int_0^{\sigma'} \mathrm{d}\sigma'\, H_{t-\sigma}(\xi_{[123]}) H_{\sigma-\sigma'}(\xi_{[12]}) H_{\sigma'-s_1}(\xi_1) H_{\sigma'-s_2}(\xi_2) H_{\sigma-s_3}(\xi_3)$$

and ...

... and

$$G^{\text{[tree]}}(t,x,\eta_{1234}) = e^{i\xi_{[1234]}x}\int_0^t \mathrm{d}\sigma \int_0^{\sigma'} \mathrm{d}\sigma' \int_0^{\sigma'} \mathrm{d}\sigma''\, H_{t-\sigma}(\xi_{[1234]}) H_{\sigma-s_4}(\xi_4) H_{\sigma-\sigma'}(\xi_{[123]}) \times$$

$$\times H_{\sigma'-s_3}(\xi_3) H_{\sigma'-\sigma''}(\xi_{[12]}) H_{\sigma''-s_1}(\xi_1) H_{\sigma''-s_2}(\xi_2)$$

and

$$G^{\text{[tree]}}(t,x,\eta_{1234}) = e^{i\xi_{[1234]}x}\int_0^t \mathrm{d}\sigma \int_0^{\sigma'} \mathrm{d}\sigma' \int_0^{\sigma} \mathrm{d}\sigma''\, H_{t-\sigma}(\xi_{[1234]}) H_{\sigma-\sigma''}(\xi_{[34]}) H_{\sigma-\sigma'}(\xi_{[12]}) \times$$

$$\times H_{\sigma'-s_1}(\xi_1) H_{\sigma'-s_2}(\xi_2) H_{\sigma''-s_3}(\xi_3) H_{\sigma''-s_4}(\xi_4)$$

and so on: you get the idea...

▷ The general explicit formula for the chaos decomposition of a polynomial

$$\int_{(\mathbb{R}\times E)^n} f(\eta_{1\cdots n}) \prod_{i=1}^{n} W(\mathrm{d}\eta_i)$$

is given by

$$\int_{(\mathbb{R}\times E)^n} f(\eta_{1\cdots n}) \prod_{i=1}^{n} W(\mathrm{d}\eta_i) = \sum_{k=0}^{n} \int_{(\mathbb{R}\times E)^k} f_k(\eta_{1\cdots k}) W(\mathrm{d}\eta_{1\cdots k})$$

with $f_k(\eta_{1\cdots k}) = 0$ if $n-k$ is odd and if $n-k = 2m$ for some m then

$$f_k(\eta_{1\cdots k}) = \sum_{\sigma\in\mathcal{S}_n} \int_{(\mathbb{R}\times E)^m} f(\sigma\eta_{1\cdots n}) \mathrm{d}\eta_{(k+1)\cdots(k+m)}$$

with the understanding that $\eta_{k+m+l} = \eta_{-(k+l)}$ for $l = 1, \ldots, m$ and where $\sigma\eta_{1\cdots n} = \eta_{\sigma(1)\cdots\sigma(n)}$.

▷ For example:

$$W(\mathrm{d}\eta_1)W(\mathrm{d}\eta_2) = W(\mathrm{d}\eta_1\mathrm{d}\eta_2) + \delta(\eta_1 + \eta_{-2})\mathrm{d}\eta_1\mathrm{d}\eta_2.$$

▷ In general we will denote with G_k^τ the kernel of the n-th chaos arising from the decomposition of X^τ:

$$X^\tau(t,x) = \sum_{k=0}^{n} \int_{(\mathbb{R}\times E)^k} G_k^\tau(t,x,\eta_{1\cdots k}) W(\mathrm{d}\eta_{1\cdots k}).$$

▷ Terms X^τ of odd degree have zero mean by construction while the terms of even degree have zero mean due to the fact that if $d(\tau)=2n$ we have

$$\mathbb{E}[X^\tau(t,x)] = \sum_{\sigma\in\mathcal{S}_{2n}} \int_{(\mathbb{R}\times E)^n} G^\tau(t,x,\sigma(\eta_{1\cdots n(-1)\cdots(-n)}))\mathrm{d}\eta_{1\ldots n}$$

where $\sigma(\eta_{1\cdots(2n)}) = \eta_{\sigma(1)\cdots\sigma(2n)}$. But now $\xi_{[1\cdots n(-1)\cdots(-n)]} = \xi_1+\cdots+\xi_n-\xi_1\cdots-\xi_n = 0$ and we always have $G^\tau(t,x,\eta_{1\cdots 2n}) \propto \xi_{[1\cdots(2n)]}$ which implies that

$$G^\tau(t,x,\sigma(\eta_{1\cdots n(-1)\cdots(-n)})) = 0.$$

This is a simplification of SBE with respect to KPZ.

Wick contractions

▷ Applying these considerations to the first nontrivial case given by $X^{\vee}$ we obtain:

$$X^{\vee}(t,x) = \int_{(\mathbb{R}\times E)^2} G^{\vee}(t,x,\eta_{12}) W(\mathrm{d}\eta_1 \mathrm{d}\eta_2) + G_0^{\vee}(t,x)$$

with

$$G_0^{\vee}(t,x) = \int_{(\mathbb{R}\times E)^2} G^{\vee}(t,x,\eta_{1(-1)}) \mathrm{d}\eta_1$$

but as already remarked

$$G^{\vee}(t,x,\eta_{1(-1)}) = e^{i\xi_{[1(-1)]}x} \int_0^t H_{t-\sigma}(\xi_{[1(-1)]}) H_{\sigma-s_1}(\xi_1) H_{\sigma-s_2}(\xi_{-1}) \mathrm{d}\sigma = 0$$

since $H_{t-\sigma}(0) = 0$.

Consider the next term

$$X^{\curlyvee}(t,x) = \int_{(\mathbb{R}\times E)^3} G^{\curlyvee}(t,x,\eta_{123}) W(\mathrm{d}\eta_1 \mathrm{d}\eta_2 \mathrm{d}\eta_3) + \int_{\mathbb{R}\times E} G_1^{\curlyvee}(t,x,\eta_1) W(\mathrm{d}\eta_1)$$

in this case we have three possible contractions contributing to $G_1^{\curlyvee}$ which results in

$$G_1^{\curlyvee}(t,x,\eta_1) = \int_{\mathbb{R}\times E} (G^{\curlyvee}(t,x,\eta_{12(-2)}) + G^{\curlyvee}(t,x,\eta_{21(-2)}) + G^{\curlyvee}(t,x,\eta_{2(-2)1}))\mathrm{d}\eta_2,$$

but note that $G^{\curlyvee}(t,x,\eta_{2(-2)1}) = 0$ since, as above, this kernel is proportional to $\xi_{2(-2)} = 0$, moreover by symmetry $G^{\curlyvee}(t,x,\eta_{12(-2)}) = G^{\curlyvee}(t,x,\eta_{21(-2)})$ so we remains with

$$G_1^{\curlyvee}(t,x,\eta_1) = G^{\mho}(t,x,\eta_1) = \int_{\mathbb{R}\times E} G^{\curlyvee}(t,x,\eta_{12(-2)})\mathrm{d}\eta_2$$

where we introduced the intuitive notation $G^{\mho}(t,x,\eta_1)$ which is useful to keep track graphically of the Wick contration on the structure of the kernels G^{τ} by representing them as arcs between leaves of the binary tree.

Wick contractions (cont.)

▷ Now an easy computation gives

$$G^{\vee}(t,x,\eta_1) = e^{i\xi_1 x}\int_0^t \mathrm{d}\sigma \int_0^\sigma \mathrm{d}\sigma' H_{t-\sigma}(\xi_1) H_{\sigma'-s_1}(\xi_1) V^{\vee}(\sigma-\sigma',\xi_1)$$

where

$$V^{\vee}(\sigma,\xi_1) = 2\int H_\sigma(\xi_{[1(-2)]}) H_{\sigma-s_2}(\xi_2) H_{-s_2}(\xi_{-2}) \mathrm{d}\eta_2 = 2\int \mathrm{d}\xi_2 H_\sigma(\xi_{[1(-2)]}) \frac{e^{-|\sigma|\xi_2^2}}{2}.$$

We call the functions V_n^τ vertex functions they are useful to compare the behaviour of different kernels.

By similar arguments we can estabilish the decomposition for the last two terms: that is

$$X^{\tau_1}(t,x)=\int_{(\mathbb{R}\times E)^3} G^{\tau_1}(t,x,\eta_{1234})W(\mathrm{d}\eta_{1234})+\int_{(\mathbb{R}\times E)^2} G_2^{\tau_1}(t,x,\eta_{12})W(\mathrm{d}\eta_{12})$$

and

$$X^{\tau_2}(t,x)=\int_{(\mathbb{R}\times E)^3} G^{\tau_2}(t,x,\eta_{1234})W(\mathrm{d}\eta_{1234})+\int_{(\mathbb{R}\times E)^2} G_2^{\tau_2}(t,x,\eta_{12})W(\mathrm{d}\eta_{12})$$

with

$$G_2^{\tau_1}(t,x,\eta_{12})=\int_{\mathbb{R}\times E} (G^{\tau_1}(t,x,\eta_{123(-3)})+2G^{\tau_1}(t,x,\eta_{132(-3)})+2G^{\tau_1}(t,x,\eta_{312(-3)}))\mathrm{d}\eta_3$$

$$=G^{\tau_3}(t,x,\eta_{12})+G^{\tau_4}(t,x,\eta_{12})+G^{\tau_5}(t,x,\eta_{12})$$

and

$$G_2^{\tau_2}(t,x,\eta_{12})=4\int_{\mathbb{R}\times E} G^{\tau_7}(t,x,\eta_{132(-3)})\mathrm{d}\eta_3=G^{\tau_6}(t,x,\eta_{12}).$$

▷ Here the contributions associated to $G^{\curlyvee}(t,x,\eta_{12})$ and $G^{\curlyvee}(t,x,\eta_{12})$ are "reducible" since they can be conveniently factorized as follows

$$G^{\curlyvee}(t,x,\eta_{12}) = \int_{\mathbb{R}\times E} G^{\curlyvee}(t,x,\eta_{123(-3)})\mathrm{d}\eta_3$$

$$= e^{i\xi_{[12]}x}\int_0^t \mathrm{d}\sigma \int_0^{\sigma} \mathrm{d}\sigma' \int_0^{\sigma'} \mathrm{d}\sigma''\, H_{t-\sigma}(\xi_{[12]})H_{\sigma'-\sigma''}(\xi_{[12]})H_{\sigma''-s_1}(\xi_1)H_{\sigma''-s_2}(\xi_2)V^{\curlyvee}(\sigma-\sigma',\xi_{[12]})$$

and

$$G^{\curlyvee}(t,x,\eta_{12}) = 2\int_{\mathbb{R}\times E} G^{\curlyvee}(t,x,\eta_{13(-3)2})\mathrm{d}\eta_3$$

$$= e^{i\xi_{[12]}x}\int_0^t \mathrm{d}\sigma H_{t-\sigma}(\xi_{[12]})H_{\sigma-s_2}(\xi_2)\int_0^{\sigma} \mathrm{d}\sigma' \int_0^{\sigma'} \mathrm{d}\sigma'' H_{\sigma-\sigma'}(\xi_1)H_{\sigma''-s_1}(\xi_1)V^{\curlyvee}(\sigma'-\sigma'',\xi_{[12]})$$

$$= e^{i\xi_{[12]}x}\int_0^t \mathrm{d}\sigma H_{t-\sigma}(\xi_{[12]})H_{\sigma-s_2}(\xi_2)e^{-\xi_1 x}G_1^{\curlyvee}(\sigma,x,\eta_1)$$

▷ $G^{\vartheta}(t,x,\eta_{12})$ cannot be reduced to a form involving V^{ϑ} and instead we have for it:

$$G^{\vartheta}(t,x,\eta_{12}) = 2\int_{\mathbb{R}\times E} G^{\Upsilon}(t,x,\eta_{132(-3)})\mathrm{d}\eta_3$$

$$= e^{i\xi_{[12]}x}\int_0^t \mathrm{d}\sigma\int_0^{\sigma}\mathrm{d}\sigma'\int_0^{\sigma'}\mathrm{d}\sigma''\, H_{t-\sigma}(\xi_{[12]})H_{\sigma'-s_2}(\xi_2)H_{\sigma''-s_1}(\xi_1)V^{\vartheta}(\sigma-\sigma',\sigma-\sigma'',\xi_{12})$$

with

$$V^{\vartheta}(\sigma-\sigma',\sigma-\sigma'',\xi_{12}) = 2\int_E \mathrm{d}\xi_3\, H_{\sigma-\sigma'}(\xi_{[132]})H_{\sigma'-\sigma''}(\xi_{[13]})\frac{e^{-\xi_3^2|\sigma-\sigma''|}}{2}$$

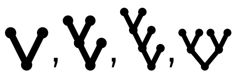
Similarly for $G^{\bowtie}$ we have

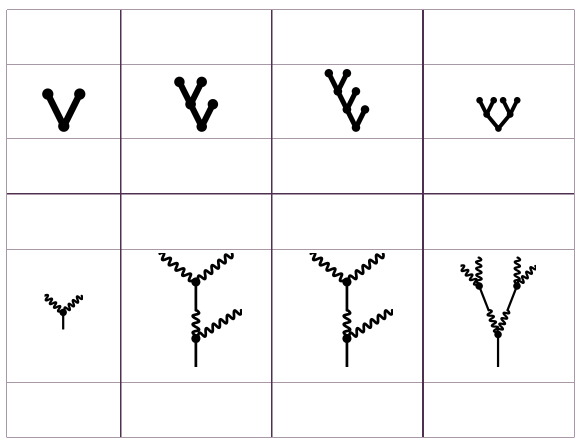
$$G^{\bowtie}(t,x,\eta_{12}) = 4\int_{\mathbb{R}\times E} G^{\curlyvee\curlyvee}(t,x,\eta_{132(-3)})\mathrm{d}\eta_3$$

$$= e^{i\xi_{[12]}x}\int_0^t \mathrm{d}\sigma\int_0^{\sigma}\mathrm{d}\sigma'\int_0^{\sigma}\mathrm{d}\sigma''\, H_{t-\sigma}(\xi_{[12]})H_{\sigma''-s_1}(\xi_1)H_{\sigma'-s_2}(\xi_2)V^{\bowtie}(\sigma-\sigma',\sigma-\sigma'',\xi_{12})$$

Feynman diagrams

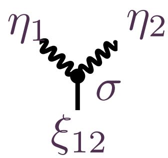
▷ The explicit form of the kernels G^τ can be described in terms of Feynman diagrams and the associated rules. To each kernel G^τ we can associate a graph which is isomorphic to the tree τ and this graph can be mapped with Feynman rules to the explicit functional form of G^τ. The algorithm goes as follows: consider τ as a graph where each edge and each internal vertex (i.e. not a leaf) are drawn as

▷ To the trees  we associate, respectively, the diagrams



▷ These diagrams corresponds to kernels via the following rules:

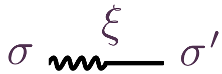
Each internal vertex comes with an time integration and a factor $(i\xi)$,



$$\longrightarrow \qquad (i\xi_{12})\int_{\mathbb{R}} \mathrm{d}\sigma$$

Each external wiggly line is associated to a variable η_i and a factor of $H_{\sigma - s_i}(\xi_i)$ where σ is the integration variable of the internal vertex to which the line is attached.

Response lines:

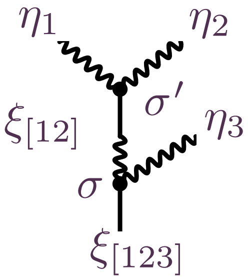


$$\longrightarrow \qquad h_{\sigma-\sigma'}(\xi)$$

Note that these lines carry information about the casual propagation.

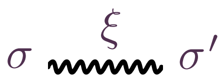
Finally the outgoing line always carries a factor $h_{t-\sigma}(\xi)$ where ξ is the outgoing momentum and σ the time label of the vertex to which the line is attached.

- For example:



$$\longrightarrow \int_{\mathbb{R}^2} \mathrm{d}\sigma \mathrm{d}\sigma' (i\xi_{[123]})(i\xi_{[12]}) h_{t-\sigma}(\xi_{[123]}) h_{\sigma-\sigma'}(\xi_{[12]}) \times \\ \times H_{\sigma'-s_1}(\xi_1) H_{\sigma'-s_2}(\xi_2) H_{\sigma-s_3}(\xi_3)$$

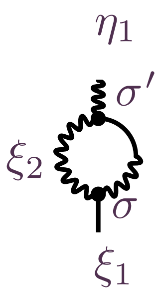
▷ Once given a diagram the associated Wick contraction are obtained by all possible pairings of the wiggly lines. To each of these pairings we associate the corresponding correlation function of the field X and an integration over the momentum variable carried by the line:



$$\longrightarrow \quad \int_E \mathrm{d}\xi \, \frac{e^{-\xi^2|\sigma-\sigma'|}}{2}$$

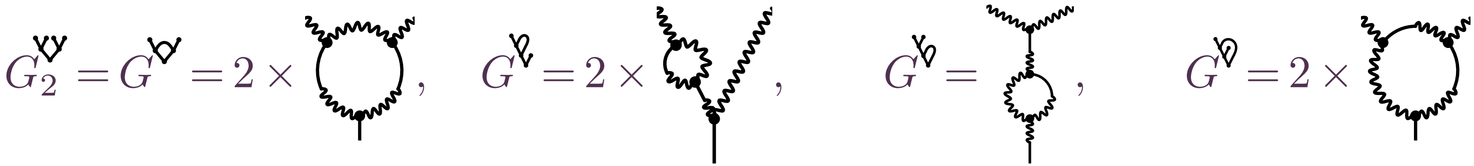
▷ For example we have :

$$G_1^{\curlyvee}(t,x,\eta_1) = 2\times$$

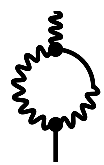


$$= 2\int \mathrm{d}\sigma \mathrm{d}\sigma' H_{t-\sigma}(\xi_1) \int_E \mathrm{d}\xi_2 \, \frac{e^{-\xi_2^2|\sigma-\sigma'|}}{2} H_{\sigma-\sigma'}\big(\xi_{[1(-2)]}\big) H_{\sigma'-s_1}(\xi_1)$$

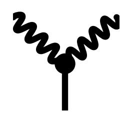
▷ Contraction arising from G and G results in the following set of diagrams:



▷ The diagrammatic representation make pictorially evident what we already have remarked with explicit computations: G and G are formed by the union of two graphs:



and



while the kernel G cannot be decomposed in such a way and it has a shape very similar to that of G.

Regularity of the driving terms

▷ Using Feymann diagrams we can compute quantities like

$$\mathbb{E}[(\Delta_q X^{\tau}(t,x))^2]$$

and obtain the pathwise regularity of the driving terms:

X	$X^{\vee}$	$X^{\vee\!\vee}$	$X^{\vee\!\vee\!\vee}$	$X^{\vee\vee}$
$-1/2-$	$0-$	$1/2-$	$1/2-$	$1-$

▷ Note that in general $B(X, f)$ for f very regular **cannot** be better than $\mathscr{C}^{1/2-}$ so we cannot hope that higher order terms in the expansion get very regular. In particular for all τ we have

$$X^{(\bullet\tau)} \in \mathscr{C}^{1/2-}$$

▷ Recall our partial expansion for the solution

$$u = X + X^{\vee} + 2\,X^{\Psi} + U$$

$$LU = 2\partial_x(U\,X) + 2\partial_x(X^{\Psi}\,X) + \partial_x(X^{\vee}X^{\vee}) + 2\partial_x((2\,X^{\Psi} + U)\,X^{\vee}) + \partial_x(2\,X^{\Psi} + U)^2$$

$$LU = 2\partial_x(U\,X) + L(2\,X^{\Psi'} + X^{\vee\vee}) + 2\partial_x((2\,X^{\Psi} + U)\,X^{\vee}) + \partial_x(2\,X^{\Psi} + U)^2$$

and the regularities for the driving terms

X	$X^{\vee}$	X^{Ψ}	$X^{\Psi'}$	$X^{\vee\vee}$
$-1/2-$	$0-$	$1/2-$	$1/2-$	$1-$

We can assume $U \in \mathscr{C}^{1/2-}$ so that the terms $2\partial_x((2\,X^{\Psi} + U)\,X^{\vee}) + \partial_x(2\,X^{\Psi} + U)^2$ are well defined.

The remaining problem is to deal with $2\partial_x(U\,X)$.

▷ Make the following ansatz $U = U' \prec Y + U^\sharp$. Then

$$LU = LU' \prec Y + U' \prec LY - \partial_x U' \prec \partial_x Y + LU^\sharp$$

while

$$LU = 2\partial_x(U X) + \underbrace{L(2 X^{[[[\bullet\bullet]\bullet]\bullet]} + X^{[[\bullet\bullet][\bullet\bullet]]}) + 2\partial_x((2 X^{[[\bullet\bullet]\bullet]} + U) X^{[\bullet\bullet]}) + \partial_x(2 X^{[[\bullet\bullet]\bullet]} + U)^2}_{Q(U)}$$

$$= 2\partial_x(U \prec X) + 2\partial_x(U \circ X) + 2\partial_x(U \succ X) + Q(U)$$

$$= 2(U \prec \partial_x X) + 2(\partial_x U \prec X) + 2\partial_x(U \circ X) + 2\partial_x(U \succ X) + Q(U)$$

so we can set $U' = 2U$ and $LY = \partial_x X$ and get the equation

$$LU^\sharp = -LU' \prec Y + \partial_x U' \prec \partial_x Y + 2(\partial_x U \prec X) + 2\partial_x(U \circ X) + 2\partial_x(U \succ X) + Q(U)$$

▷ Observe that $Y, U, U' \in \mathscr{C}^{1/2-}$ and we can assume that $U^\sharp \in \mathscr{C}^{1-}$.

The resonant term

▷ The difficulty is now concentrated in the resonant term $U \circ X$ which is not well defined.

▷ The paracontrolled ansatz and the commutation lemma give

$$U \circ X = (2U \prec Y) \circ X + U^\sharp \circ X = 2\,U(Y \circ X) + \underbrace{C(2\,U, Y, X)}_{1/2-} + \underbrace{U^\sharp \circ X}_{1/2-}$$

▷ A stochastic estimate shows that $Y \circ X \in \mathscr{C}^{0-}$

▷ The final fixed point equation reads

$$LU^\sharp = 4\partial_x\,(U(Y \circ X)) + 4\,\partial_x C(U, Y, X) + 2\partial_x(U^\sharp \circ X) - 2\,LU \prec Y$$

$$+2\partial_x U \prec \partial_x Y + 2(\partial_x U \prec X) + 2\partial_x(U \succ X) + Q(U)$$

▷ This equation has a (local in time) solution $U = \Phi(\mathbb{X}(\xi))$ which is a continuous function of the data $\mathbb{X}(\xi)$ given by the collection of multilinear functions of ξ given by

$$\mathbb{X}(\xi) = (X, X^{\vee}, X^{\vee\!\!\vee}, X^{\vee\!\!\vee}, X^{\vee\vee}, X \circ Y)$$

Thanks.