Combinatorial Dyson-Schwinger equations and systems III
systems of Dyson-Schwinger equations

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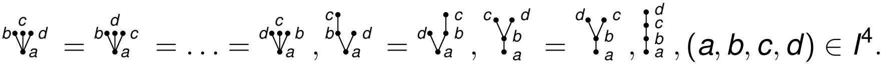
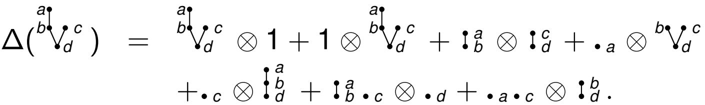
Potsdam
November 2013

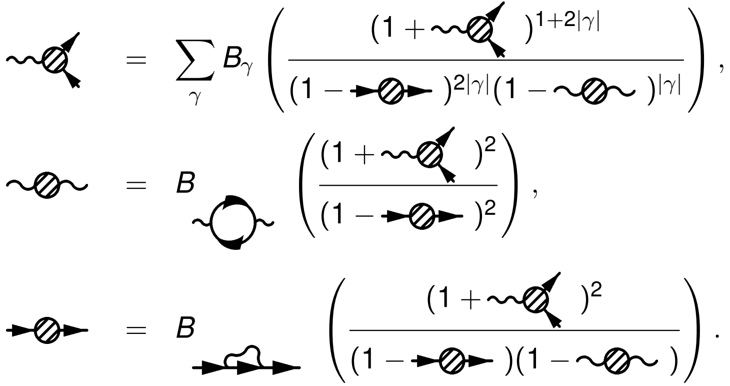
Let I be a set. Rooted trees decorated by I:



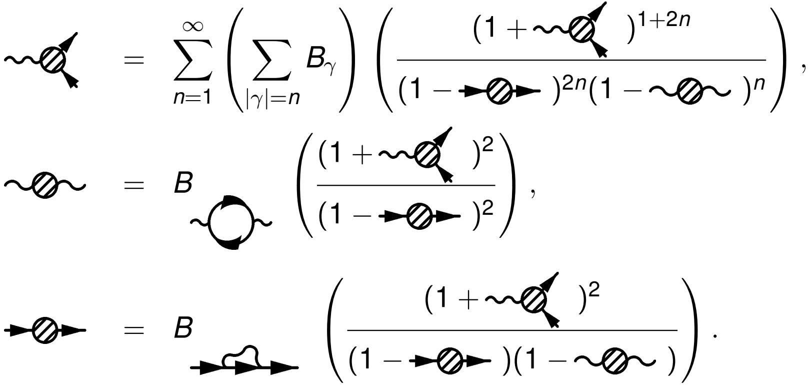
Coproduct:



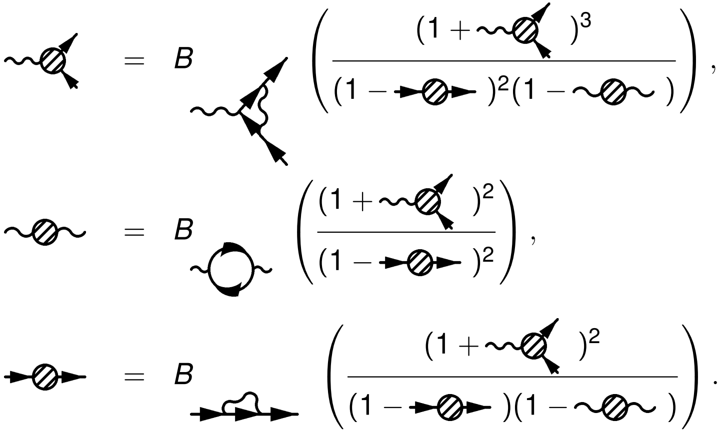
Dyson-Schwinger system from QED:



Dyson-Schwinger system from QED:

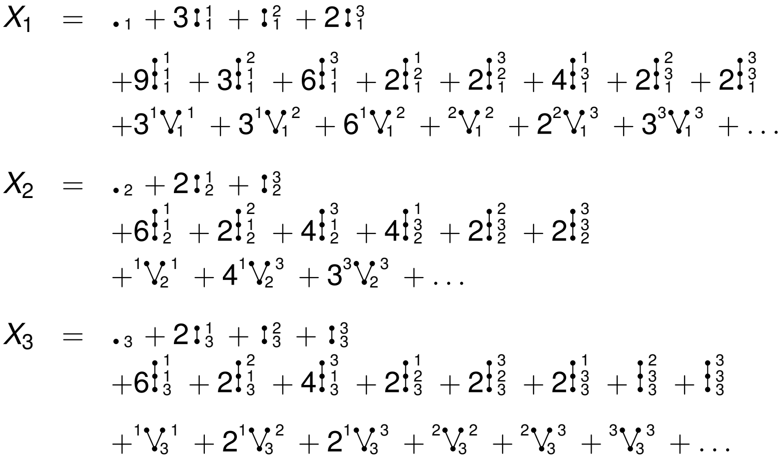


Dyson-Schwinger system from QED truncated at order 1:



Lifting to decorated trees:

$$\begin{aligned} X_1 &= B_1\left(\frac{(1+X_1)^3}{(1-X_3)^2(1-X_2)}\right), \\ X_2 &= B_2\left(\frac{(1+X_1)^2}{(1-X_3)^2}\right), \\ X_3 &= B_3\left(\frac{(1+X_1)^2}{(1-X_2)(1-X_3)}\right). \end{aligned}$$



Definition

- Let $f_1, \ldots, f_n \in K[[h_1, \ldots, h_n]] - K$. The combinatorial Dyson-Schwinger systems attached to $f = (f_1, \ldots, f_n)$ is:

$$(S): \left\{ \begin{array}{rcl} X_1 & = & B_1^+(f_1(X_1, \ldots, X_n)) \\ & \vdots & \\ X_n & = & B_n^+(f_n(X_1, \ldots, X_n)), \end{array} \right.$$

- Such a system has a unique solution $(X_1, \ldots, X_n) \in \widehat{H_R^{\{1,\ldots,n\}}}$.
- The subalgebra generated by the homogeneous components of the $X(i)$'s is denoted by $H_{(S)}$.
- If this subalgebra is Hopf, we shall say that the system is Hopf.

Graph associated to (S)

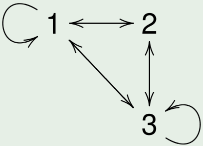
Let (S) be associated to $(f_1, \ldots, f_n)$. The oriented graph associated to (S) is defined by:

1. The vertices are $1, \ldots, n$.
2. There is an edge from i to j if, and only if, $\dfrac{\partial f_i}{\partial h_j} \neq 0$.

Example coming from QED

$$\begin{aligned} X_1 &= B_1\left(\frac{(1+X_1)^3}{(1-X_3)^2(1-X_2)}\right), \\ X_2 &= B_2\left(\frac{(1+X_1)^2}{(1-X_3)^2}\right), \\ X_3 &= B_3\left(\frac{(1+X_1)^2}{(1-X_2)(1-X_3)}\right). \end{aligned}$$

Graph:



Change of variables

Let (S) be the following system:

$$(S): \left\{ \begin{array}{rcl} X_1 & = & B_1^+(f_1(X_1, \ldots, X_n)) \\ & \vdots & \\ X_n & = & B_n^+(f_n(X_1, \ldots, X_n)). \end{array} \right.$$

If (S) is Hopf, then for all family $(\lambda_1, \ldots, \lambda_n)$ of non-zero scalars, this system is Hopf:

$$(S): \left\{ \begin{array}{rcl} X_1 & = & B_1^+(f_1(\lambda_1 X_1, \ldots, \lambda_n X_n)) \\ & \vdots & \\ X_n & = & B_n^+(f_n(\lambda_1 X_1, \ldots, \lambda_n X_n)). \end{array} \right.$$

Concatenation

Let (S) and (S') be the following systems:

$$(S):\left\{\begin{array}{rcl} X_1 & = & B_1^+(f_1(X_1,\ldots,X_n)) \\ & \vdots & \\ X_n & = & B_n^+(f_n(X_1,\ldots,X_n)). \end{array}\right.$$

$$(S'):\left\{\begin{array}{rcl} X_1 & = & B_1^+(g_1(X_1,\ldots,X_m)) \\ & \vdots & \\ X_m & = & B_m^+(g_m(X_1,\ldots,X_m)). \end{array}\right.$$

Concatenation

The following system is Hopf if, and only if, the (S) and (S') are Hopf:

$$\begin{cases} X_1 = B_1^+(f_1(X_1,\ldots,X_n)) \\ \vdots \\ X_n = B_n^+(f_n(X_1,\ldots,X_n)) \\ X_{n+1} = B_{n+1}^+(g_1(X_{n+1},\ldots,X_{n+m})) \\ \vdots \\ X_{n+m} = B_{n+m}^+(g_m(X_{n+1},\ldots,X_{n+m})). \end{cases}$$

This property leads to the notion of connected (or indecomposable) system.

Extension

Let (S) be the following system:

$$(S) : \begin{cases} X_1 = B_1^+(f_1(X_1, \ldots, X_n)) \\ \vdots \\ X_n = B_n^+(f_n(X_1, \ldots, X_n)). \end{cases}$$

Then (S') is an extension of (S):

$$(S') : \begin{cases} X_1 = B_1^+(f_1(X_1, \ldots, X_n)) \\ \vdots \\ X_n = B_n^+(f_n(X_1, \ldots, X_n)) \\ X_{n+1} = B_{n+1}^+(1 + a_1 X_1). \end{cases}$$

Iterated extensions

$$(S):\left\{\begin{array}{rcl} X_1 &=& B_1\left((1-\beta X_1)^{-\frac{1}{\beta}}\right), \\ X_2 &=& B_2(1+X_1), \\ X_3 &=& B_3(1+X_1), \\ X_4 &=& B_4(1+2X_2-X_3), \\ X_5 &=& B_5(1+X_4). \end{array}\right.$$

Dilatation

(S') is a dilatation of (S):

$$(S):\left\{\begin{array}{rcl} X_1 &=& B_1^+(f(X_1,X_2)),\\ X_2 &=& B_2^+(g(X_1,X_2)),\end{array}\right.$$

$$(S'):\left\{\begin{array}{rcl} X_1 &=& B_1^+(f(X_1+X_2+X_3,X_4+X_5)),\\ X_2 &=& B_2^+(f(X_1+X_2+X_3,X_4+X_5)),\\ X_3 &=& B_3^+(f(X_1+X_2+X_3,X_4+X_5)),\\ X_4 &=& B_4^+(g(X_1+X_2+X_3,X_4+X_5)),\\ X_5 &=& B_5^+(g(X_1+X_2+X_3,X_4+X_5)).\end{array}\right.$$

Fundamental systems

Let $\beta_1, \ldots, \beta_k \in K$. The following system is an example of a *fundamental* system:

$$\begin{cases} X_i = B_i\left((1-\beta_i X_i)\displaystyle\prod_{j=1}^{k}(1-\beta_j X_j)^{-\frac{1+\beta_j}{\beta_j}} \prod_{j=k+1}^{n}(1-X_j)^{-1}\right) & \text{if } i \leq k, \\ X_i = B_i\left((1-X_i)\displaystyle\prod_{j=1}^{k}(1-\beta_j X_j)^{-\frac{1+\beta_j}{\beta_j}} \prod_{j=k+1}^{n}(1-X_j)^{-1}\right) & \text{if } i > k. \end{cases}$$

Cyclic systems

The following systems are *cyclic*: if $n \geq 2$,

$$\begin{cases} X_1 &= B_1^+(1+X_2), \\ X_2 &= B_2^+(1+X_3), \\ &\vdots \\ X_n &= B_n^+(1+X_1). \end{cases}$$

Graph on a cyclic system: an oriented cycle.

Theorem

Let (S) be an SDSE. If it is Hopf, then, for all $i, j \in I$, for all $n \geq 1$, there exists a scalar $\lambda_n^{(i,j)}$ such that for all tree t', which root is decorated by i:

$$\sum_t n_j(t, t') a_t = \lambda_{|t'|}^{(i,j)} a_{t'},$$

where $n_j(t, t')$ is the number of leaves ℓ of t decorated by j such that the cut of ℓ gives t'.

We shall denote by $a_j^{(i)}$ the coefficient of h_j in f_i and by $a_{j,k}^{(i)}$ the coefficient of $h_j h_k$ in f_i.

Lemma

$\frac{\partial f_i}{\partial h_j} \neq 0$ if, and only if, $a_j^{(i)} \neq 0$.

Theorem

Let us assume that (S) is Hopf. Let us fix i.

1. For all path $i = i_1 \to i_2 \to \ldots \to i_k$ in the graph of (S)

$$\lambda_k^{(i,j)} = a_j^{(i_k)} + \sum_{p=1}^{k-1} (1 + \delta_{j,i_{p+1}}) \frac{a_{j,i_{p+1}}^{(i_p)}}{a_{i_{p+1}}^{(i_p)}}.$$

   In particular, $\lambda_1^{(i,j)} = a_j^{(i)}$.

2. For all $p_1, \cdots, p_n \in \mathbb{N}$:

$$a^{(i)}_{(p_1,\cdots,p_j+1,\cdots,p_n)} = \frac{1}{p_j+1} \left(\lambda^{(i,j)}_{p_1+\cdots+p_n+1} - \sum_{l \in I} p_l a_j^{(l)} \right) a^{(i)}_{(p_1,\cdots,p_n)}.$$

Lemma

Let (S) be a Hopf SDSE. In the graph associated to (S):

$$\begin{array}{ccc} i & \longrightarrow & j \\ \downarrow & & \downarrow \\ l & & k \end{array} \quad \Longrightarrow \quad \begin{array}{ccc} i & \longrightarrow & j \\ \downarrow & & \downarrow \\ l & \longrightarrow & k \end{array} \quad \text{or} \quad \begin{array}{ccc} i & \longrightarrow & j \\ \downarrow & \searrow & \downarrow \\ l & & k \end{array}.$$

Let us assume that $a_k^{(i)} = 0$. As $a_j^{(i)} \neq 0$, $j \neq k$. As $a_k^{(i)} = 0$,

$$a_{{}^{j}\bigvee_{i}{}^{k}} = a_{j,k}^{(i)} = 0.$$

Then:

$$\lambda_2^{(i,k)} a_j^{(i)} = \lambda_2^{(i,k)} a_{\mathbf{1}^{j}_{i}} = a_{\mathbf{1}^{k}_{j\,i}} + a_{{}^{j}\bigvee_{i}{}^{k}} = a_j^{(i)} a_k^{(j)} + 0;$$

Hence:

$$\lambda_2^{(i,k)} = a_k^{(j)} \neq 0.$$

Moreover, As $a_l^{(i)} \neq 0$, $l \neq k$. Then:

$$a_l^{(i)} \lambda_2^{(i,k)} = \lambda_2^{(i,k)} a_{\substack{l\\ i}} = a_{\substack{k\\ l\\ i}} + a_{\bigvee_i^{l\,k}} = a_l^{(i)} a_k^{(l)} + 0.$$

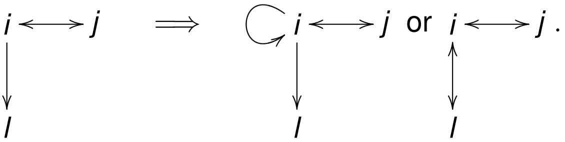
so:

$$\lambda_2^{(i,k)} = a_k^{(l)}.$$

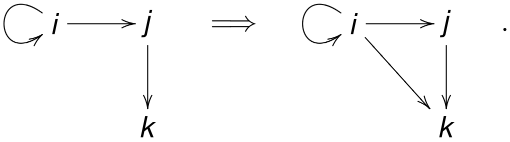
Hence:

$$a_k^{(l)} = a_k^{(j)} \neq 0.$$

1. A first special case is given by $i = k$:



2. A second special case is given by $i = l$:



Proposition

Let (S) be a Hopf Dyson-Schwinger system with the following graph:

$$1 \longleftrightarrow 2\,.$$

Up to a change of variables, two cases can occur:

1. $(S): \left\{ \begin{array}{rcl} X_1 &=& B_1(1+X_2), \\ X_2 &=& B_2(1+X_1). \end{array} \right.$
2. $(S): \left\{ \begin{array}{rcl} X_1 &=& B_1((1-X_2)^{-1}), \\ X_2 &=& B_2((1-X_1)^{-1}). \end{array} \right.$

We put:

$$f_1(h_2)=\sum_{i=0}^{\infty} a_i h_2^i, \qquad f_2(h_1)=\sum_{i=0}^{\infty} b_i h_1^i.$$

Up to a change of variables, assume that $a_1=b_1=1$. Then:

$$\lambda_3^{(1,1)}=\lambda_3^{(1,1)} a_{\substack{1\\2\\1}}=2a_{{}^{1}\mathrm{V}^{1}_{2,1}}=2b_2.$$

On the other hand:

$$2a_2b_2=\lambda_3^{(1,1)} a_{{}_2\mathrm{V}^{2}_{1}}=a_{{}^{2}\mathrm{V}^{1,2}_{1}}=2a_2.$$

So $2a_2b_2=2a_2$ and $a_2=0$ or $b_2=1$. Similarly, $b_2=0$ or $a_2=1$. Finally:

$$a_2=b_2=0 \text{ or } 1.$$

In the first case, $f_1(h_2) = 1 + h_2$ and $f_2(h_1) = 1 + h_1$. In the second case, consider the path $1 \to 2 \to 1 \to \ldots$ of length n.

- If $n = 2k$ is even:

$$\lambda_n^{(1,2)} = 2 + 2(k-1) = 2k = n.$$

- If $n = 2k+1$ is odd:

$$\lambda_n^{(1,2)} = 1 + 2k = n.$$

So:

$$\lambda_n^{(1,2)} = n \text{ for all } n \geq 1.$$

Hence, for all $n \geq 1$, $a_{n+1} = a_n$ and finally $f_1(h_2) = (1 - h_2)^{-1}$. Similarly, $f_2(h_1) = (1 - h_1)^{-1}$.

Main theorem

Let (S) be Hopf combinatorial Dyson-Schwinger system. Then (S) is obtained from the concatenation of fundamental or cyclic systems with the help of a change of variables, a dilatation and a finite number of extensions.

If (S) is a Hopf, the dual of $H_{(S)}$ is the enveloping algebra of a prelie algebra $\mathfrak{g}_{(S)}$.

Description of $\mathfrak{g}_{(S)}$

It has a basis $(e_i(p))_{1\leq i\leq n, p\geq 1}$. The prelie product is given by:

$$e_i(p) \circ e_j(q) = \lambda_q^{(j,i)} e_j(p+q).$$

As a consequence, $\mathfrak{g}_i = Vect(e_i(p), p \geq 1)$ is a prelie subalgebra. In the fundamental case, there are three possibilities:

1. $i \leq k$, with $\beta_i = -1$. Then $e_i(p) \circ e_i(q) = e_i(p+q)$: $\mathfrak{g}_i$ is an associative, commutative algebra.
2. $i > k$. Then $e_i(p) \circ e_i(q) = 0$: $\mathfrak{g}_i$ is a trivial prelie algebra.
3. $i \leq k$ and $\beta_i \neq -1$. Then $b_j \neq 0$, and $\mathfrak{g}_i$ is a Faà di Bruno prelie algebra with parameter given by:

$$\lambda_i = \frac{-\beta_i}{1+\beta_i}.$$

Proposition

Let (S) be a fundamental SDSE. If $k < n$ or if there exists $i \leq k$, such that $\beta_i \neq -1$, then the Lie algebra $\mathfrak{g}_{(S)}$ can be decomposed in a semi-direct product:

$$\mathfrak{g}_{(S)} = (M_1 \oplus \ldots \oplus M_k) \rtimes \mathfrak{g}_0,$$

where:

- $\mathfrak{g}_0$ is a Lie subalgebra of $\mathfrak{g}_{(S)}$, isomorphic to the Faà di Bruno Lie algebra, with basis $(f_n^0)_{n\geq 1}$ such that for all $m, n \geq 1$:
$$[f_m^0, f_n^0] = (n-m)f_{n+m}^0.$$
- For all $1 \leq i \leq k$, M_i is an abelian Lie subalgebra of $\mathfrak{g}_{(S)}$, with basis $(f_n^i)_{n\geq 1}$.

Proposition

- For all $1 \leq i \leq k$, M_i is a left $\mathfrak{g}_0$-module in the following way:

$$f_m^0.f_n^i = nf_{m+n}^i.$$

Proposition

Let (S) be a cyclic SDSE, possibly with dilatations and extensions. The prelie $\mathfrak{g}_{(S)}$ admits a basis $(e_i(k))_{1\leq i\leq n, k\geq 1}$ such that:

$$e_i(k)\circ e_j(l) = \begin{cases} e_j(k+l) \text{ if there exists a path from } j \text{ to } i \text{ of length } l, \\ 0 \text{ otherwise.} \end{cases}$$

This prelie product is associative.

We now consider systems of the form :

$$(S):\left\{\begin{array}{rcl} X_1 & = & \displaystyle\sum_{i\in J_1} B^+_{1,i}(f_{1,i}(X_1,\ldots,X_n)) \\ & \vdots & \\ X_n & = & \displaystyle\sum_{i\in J_n} B^+_{n,i}(f_{n,i}(X_1,\ldots,X_n)), \end{array}\right.$$

where for all k, i, $B_{k,i}$ is a 1-cocycle of degree i.

Theorem

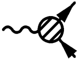
We assume that $1 \in J_k$ for all k. Then (S) is entirely determined by $f_{1,1},\ldots,f_{n,1}$.

Fundamental system

$$\begin{cases} X_i = \displaystyle\sum_{q \in J_i} B_{i,q}\left((1-\beta_i X_i)\prod_{j=1}^{k}(1-\beta_j X_j)^{-\frac{1+\beta_j}{\beta_j}q}\prod_{j=k+1}^{n}(1-X_j)^{-q}\right) & \text{if } i \leq k, \\ X_i = \displaystyle\sum_{q \in J_i} B_{i,q}\left((1-X_i)\prod_{j=1}^{k}(1-\beta_j X_j)^{-\frac{1+\beta_j}{\beta_j}q}\prod_{j=k+1}^{n}(1-X_j)^{-q}\right) & \text{if } i > k. \end{cases}$$

For example, we choose $n = 3$, $k = 2$, $\beta_1 = -1/3$ $\beta_2 = 1$, $J_1 = \mathbb{N}^*$, $J_2 = J_3 = \{1\}$. After a change of variables $h_1 \longrightarrow 3h_1$, we obtain:

$$(S) : \left\{ \begin{array}{rcl} X_1 & = & \displaystyle\sum_{k \geq 1} B_{1,k}\left(\frac{(1+X_1)^{1+2k}}{(1-X_2)^{2k}(1-X_3)^k}\right), \\ X_2 & = & B_2\left(\frac{(1+X_1)^2}{(1-X_2)(1-X_3)}\right), \\ X_3 & = & B_3\left(\frac{(1+X_1)^2}{(1-X_2)}\right). \end{array} \right.$$

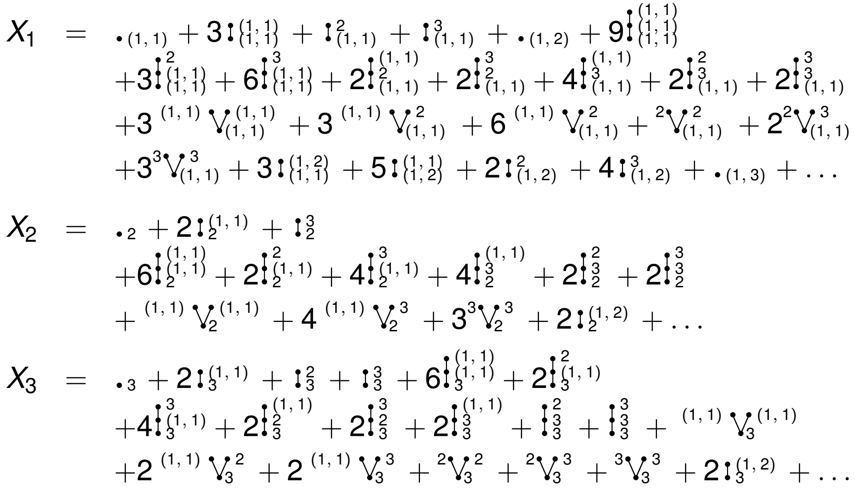
This is the example of the introduction, with $X_1 =$  ,

$X_2 =$  , $X_3 =$  .

$X_1 = \ldots$

$X_2 = \ldots$

$X_3 = \ldots$



Cyclic systems

$$(S):\left\{\begin{array}{rcl} X_{\overline{1}} & = & \displaystyle\sum_{j\in I_1} B_{1,j}\left(1+X_{\overline{1+j}}\right),\\ & \vdots & \\ X_{\overline{n}} & = & \displaystyle\sum_{j\in I_1} B_{n,j}\left(1+X_{\overline{n+j}}\right). \end{array}\right.$$

$n = 3$:

$$
\begin{aligned}
X_{\bar 1} &= \bullet_{(\bar 1,1)} + \bullet_{(\bar 1,2)} + \begin{smallmatrix}\bullet\, (\bar 2,1)\\ |\\ \bullet\, (\bar 1,1)\end{smallmatrix} + \bullet_{(\bar 1,3)} + \begin{smallmatrix}\bullet\, (\bar 3,1)\\ |\\ \bullet\, (\bar 1,2)\end{smallmatrix} + \begin{smallmatrix}\bullet\, (\bar 3,1)\\ |\\ \bullet\, (\bar 2,1)\\ |\\ \bullet\, (\bar 1,1)\end{smallmatrix},\\
X_{\bar 2} &= \bullet_{(\bar 2,1)} + \bullet_{(\bar 2,2)} + \begin{smallmatrix}\bullet\, (\bar 3,1)\\ |\\ \bullet\, (\bar 2,1)\end{smallmatrix} + \bullet_{(\bar 2,3)} + \begin{smallmatrix}\bullet\, (\bar 1,1)\\ |\\ \bullet\, (\bar 2,2)\end{smallmatrix} + \begin{smallmatrix}\bullet\, (\bar 1,1)\\ |\\ \bullet\, (\bar 3,1)\\ |\\ \bullet\, (\bar 2,1)\end{smallmatrix},\\
X_{\bar 3} &= \bullet_{(\bar 3,1)} + \bullet_{(\bar 3,2)} + \begin{smallmatrix}\bullet\, (\bar 1,1)\\ |\\ \bullet\, (\bar 3,1)\end{smallmatrix} + \bullet_{(\bar 3,3)} + \begin{smallmatrix}\bullet\, (\bar 2,1)\\ |\\ \bullet\, (\bar 3,2)\end{smallmatrix} + \begin{smallmatrix}\bullet\, (\bar 2,1)\\ |\\ \bullet\, (\bar 1,1)\\ |\\ \bullet\, (\bar 3,1)\end{smallmatrix}.
\end{aligned}
$$