Combinatorial Dyson-Schwinger equations and systems II
Dyson-Schwinger equations and Prelie algebras

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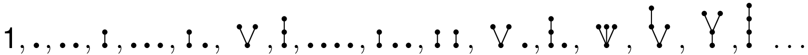
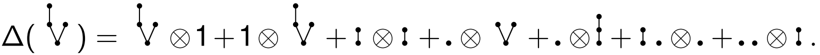
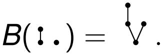
Rooted forests:



Coproduct:



Grafting operator:



Definition

Let $f(h) \in K[[h]]$.

- The combinatorial Dyson-Schwinger equations associated to $f(h)$ is:

$$X = B(f(X)),$$

where X lives in the completion of H_R.

- This equation has a unique solution $X = \sum X(n)$, with:

$$\left\{ \begin{array}{rcl} X(1) & = & p_0 \bullet, \\ X(n+1) & = & \sum_{k=1}^{n} \sum_{a_1+\ldots+a_k=n} p_k B(X(a_1)\ldots X(a_k)), \end{array} \right.$$

where $f(h) = p_0 + p_1 h + p_2 h^2 + \ldots$

$X(1) = p_0$,

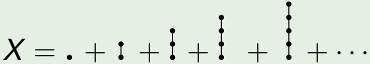
$X(2) = p_0 p_1$,

$X(3) = p_0 p_1^2$ $+ p_0^2 p_2$  ,

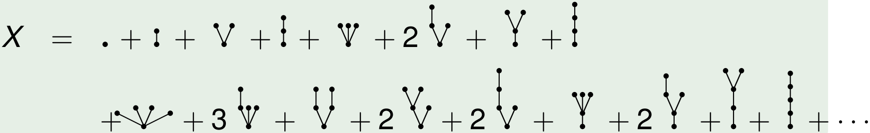
$X(4) = p_0 p_1^3$ $+ p_0^2 p_1 p_2$  $+ 2 p_0^2 p_1 p_2$  $+ p_0^3 p_3$  .

Examples

- If $f(h) = 1 + h$:



- If $f(h) = (1 - h)^{-1}$:



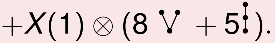
Let $f(h) \in K[[h]]$. The homogeneous components of the unique solution of the combinatorial Dyson-Schwinger equation associated to $f(h)$ generate a subalgebra of H_R denoted by H_f.

H_f is not always a Hopf subalgebra

For example, for $f(h) = 1 + h + h^2 + 2h^3 + \cdots$, then:

So:

$$\Delta(X(4)) = X(4) \otimes 1 + 1 \otimes X(4) + (10X(1)^2 + 3X(2)) \otimes X(2)$$
$$+(X(1)^3 + 2X(1)X(2) + X(3)) \otimes X(1)$$



If $f(0) = 0$, the unique solution of $X = B(f(X))$ is 0. From now, up to a normalization we shall assume that $f(0) = 1$.

Theorem

Let $f(h) \in K[[h]]$, with $f(0) = 1$. The following assertions are equivalent:

1. H_f is a Hopf subalgebra of H_R.
2. There exists $(\alpha, \beta) \in K^2$ such that $(1 - \alpha\beta h)f'(h) = \alpha f(h)$.
3. There exists $(\alpha, \beta) \in K^2$ such that $f(h) = 1$ if $\alpha = 0$ or $f(h) = e^{\alpha h}$ if $\beta = 0$ or $f(h) = (1 - \alpha\beta h)^{-\frac{1}{\beta}}$ if $\alpha\beta \neq 0$.

$1 \Longrightarrow 2$. We put $f(h) = 1 + p_1 h + p_2 h^2 + \cdots$. Then $X(1) = \bullet$.
Let us write:

$$\Delta(X(n+1)) = X(n+1) \otimes 1 + 1 \otimes X(n+1) + X(1) \otimes Y(n) + \ldots.$$

1. By definition of the coproduct, $Y(n)$ is obtained by cutting a leaf in all possible ways in $X(n+1)$. So it is a linear span of trees of degree n.
2. As H_f is a Hopf subalgebra, $Y(n)$ belongs to H_f.

Hence, there exists a scalar λ_n such that $Y(n) = \lambda_n X_n$.

lemma

Let us write:

$$X = \sum_t a_t t.$$

For any rooted tree t:

$$\lambda_{|t|} a_t = \sum_{t'} n(t,t') a_{t'},$$

where $n(t,t')$ is the number of leaves of t' such that the cut of this leaf gives t.

We here assume that f is not constant. We can prove that $p_1 \neq 0$.

For t the ladder $(B)^n(1)$, we obtain:

$$p_1^{n-1}\lambda_n = 2(n-1)p_1^{n-2}p_2 + p_1^n.$$

Hence:

$$\lambda_n = 2\frac{p_2}{p_1}(n-1) + p_1.$$

We put $\alpha = p_1$ and $\beta = 2\frac{p_2}{p_1^2} - 1$, then:

$$\lambda_n = \alpha(1 + (n-1)(1+\beta)).$$

For t the corolla $B(\bullet^{n-1})$, we obtain:

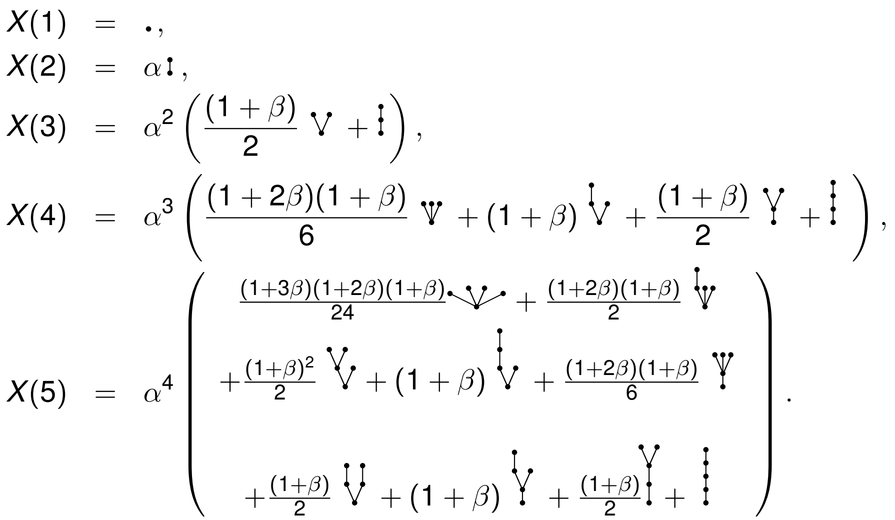
$$\lambda_n p_{n-1} = np_n + (n-1)p_{n-1}p_1.$$

Hence:

$$\alpha(1+(n-1)\beta)p_{n-1} = np_n.$$

Summing:

$$(1-\alpha\beta h)f'(h) = \alpha f(h).$$


X(1) = .,
X(2) = α,
X(3) = α^2 ((1+β)/2 +),
X(4) = α^3 ((1+2β)(1+β)/6 + (1+β) + (1+β)/2 +),
X(5) = α^4 ((1+3β)(1+2β)(1+β)/24 + (1+2β)(1+β)/2 + (1+β)^2/2 + (1+β) + (1+2β)(1+β)/6 + (1+β)/2 + (1+β) + (1+β)/2 +).

Particular cases

- If $(\alpha, \beta) = (1, -1)$, $f = 1 + h$ and $X(n) = (B)^n(1)$ for all n.
- If $(\alpha, \beta) = (1, 1)$, $f = (1 - h)^{-1}$ and:

$$X(n) = \sum_{|t|=n} \sharp\{\text{embeddings of } t \text{ in the plane}\} t.$$

- Si $(\alpha, \beta) = (1, 0)$, $f = e^h$ and:

$$X(n) = \sum_{|t|=n} \frac{1}{\sharp\{\text{symmetries of } t\}} t.$$

(Left) prelie algebra

A prelie algebra $\mathfrak{g}$ is a vector space with a linear product $\circ$ such that for all $x, y, z \in \mathfrak{g}$:

$$x \circ (y \circ z) - (x \circ y) \circ z = y \circ (x \circ z) - (y \circ x) \circ z.$$

Associated Lie bracket

If $\circ$ is a prelie product on $\mathfrak{g}$, its antisymmetrization is a Lie bracket.

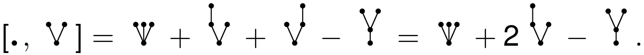
Primitive elements of the dual of H_R

For any rooted tree t let us define:

$$f_t : \begin{cases} H_R & \longrightarrow & K \\ F & \longrightarrow & s_t \delta_{F,t}. \end{cases}$$

The family (f_t) is a basis of the primitive elements of H_R^*. The Lie bracket is given by:

$$[f_{t_1}, f_{t_2}] = \sum_{t'=t_1 \rightarrowtail t_2} f_{t'} - \sum_{t'=t_2 \rightarrowtail t_1} f_{t'}.$$



We define:

$$f_{t_1} \circ f_{t_2} = \sum_{t'=t_1 \rightarrowtail t_2} f_{t'}.$$

This product is prelie.

Theorem (Chapoton-Livernet)

As a prelie algebra, $Prim(H_R^*)$ is freely generated by $f_\bullet$.

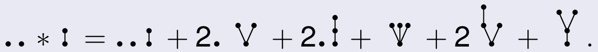
By duality with H_R, we obtain a description of the enveloping algebra of the free prelie algebra on one generators.

Grossman-Larson Hopf algebra

- Basis: the set of rooted forests.
- Coproduct :

$$\Delta(t_1 \dots t_k) = \sum_{I \subseteq \{1,\dots,k\}} \left(\prod_{i \in I} t_i\right) \otimes \left(\prod_{i \notin I} t_i\right).$$

- Product: generalized graftings.



Let $\lambda \in K$.

Faà di Bruno prelie algebra

$\mathfrak{g}_{FdB}$ has a basis $(e_i)_{i\geq 1}$, and the prelie product is defined by:

$$e_i \circ e_j = (j+\lambda)e_{i+j}.$$

For all $i,j,k \geq 1$:

$$e_i \circ (e_j \circ e_k) - (e_i \circ e_j) \circ e_k = k(k+\lambda)e_{i+j+k}.$$

Let $\mathfrak{g}$ be prelie algebra.

Theorem (Guin-Oudom)

The product $\circ$ of $\mathfrak{g}$ can be extended to $S(\mathfrak{g})$: if $a, b, c \in S_+(\mathfrak{g})$, $x \in \mathfrak{g}$,

$$\begin{cases} a \circ 1 &= \varepsilon(a), \\ 1 \circ b &= b, \\ (xa) \circ b &= x \circ (a \circ b) - (x \circ a) \circ b, \\ a \circ (bc) &= \sum (a' \circ b)(a'' \circ c). \end{cases}$$

One then defines a product on $S_+(\mathfrak{g})$ by $a \star b = \sum a'(a'' \circ b)$, with the Sweedler notation $\Delta(a) = \sum a' \otimes a''$. Then $(S(\mathfrak{g}), *, \Delta)$ is a Hopf algebra, isomorphic to the enveloping algebra of $\mathfrak{g}$.

- In $S(\mathfrak{g}_{FdB})$:

$$(e_{i_1} \ldots e_{i_m}) \circ e_j = (j+\lambda)j(j-\lambda)\ldots(j-(m-2)\lambda)e_{i_1+\ldots+i_m+j}.$$

- There exists a unique prelie algebra morphism ϕ_λ from the free prelie algebra on one generator to $\mathfrak{g}_{FdB}$, sending $\bullet$ to e_1. It is extended as a Hopf algebra morphism from $S(\mathfrak{g}_{FdB})$ to H_R^*; then by transposition we obtain a Hopf algebra morphism Φ_λ from $S(\mathfrak{g}_{FdB})^*$ to H_R.

Theorem

The image of Φ_λ is generated as an algebra by the elements $x(n) = \Phi_\lambda(e_n^*)$, $n \geq 1$. Moreover, $\sum x(n)$ is the solution of the Dyson-Schwinger equation:

$$X = B\left(\left(1 + \frac{\lambda}{1+\lambda} X\right)^{\frac{\lambda}{1+\lambda}}\right).$$

Corollary

For all $\alpha, \beta \in K$, the algebra generated by the components of the solution of the Dyson-Schwinger equation

$$X = B\left((1 - \alpha\beta X)^{-\frac{1}{\beta}}\right)$$

is a Hopf subalgebra.

Corollary

- If $\beta \neq -1$ and $\alpha = 1$,

$$\Delta(X) = X \otimes 1 + \sum_{j=1}^{\infty} (1 + \lambda X)^{1+\frac{j}{\lambda}} \otimes X(j),$$

  with $\lambda = \dfrac{-1}{1+\beta}$.
- If $\beta = -1$ and $\alpha = 1$,

$$\Delta(X) = 1 \otimes X + X \otimes 1 + X \otimes X.$$

Hence, we have a family of Hopf subalgebras $H_{(\alpha,\beta)}$ of H_R indexed by (α, β).

Theorem

- If $\alpha \neq 0$ and $\beta = -1$, $H_{(\alpha,\beta)}$ is isomorphic to the Hopf algebra of symmetric functions.
- If $\alpha \neq 0$ and $\beta \neq -1$, $H_{(\alpha,\beta)}$ is isomorphic to the Faà di Bruno Hopf algebra. In other words, $H_{(\alpha,\beta)}$ is the coordinate ring of the group of formal diffeomorphisms of the line that are tangent to the identity:

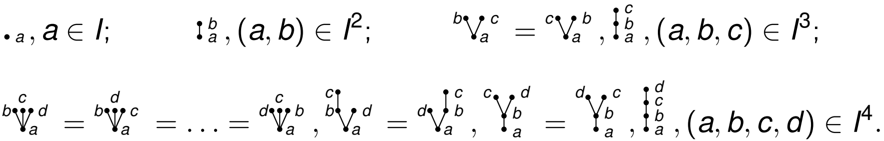
$$G = \left(\{ f(h) = h + a_1 h^2 + \ldots \mid a_1, a_2, \ldots \in K \}, \circ \right).$$

In QFT, generally Dyson-Schwinger equations involve several 1-cocycles, for example [Bergbauer-Kreimer]:

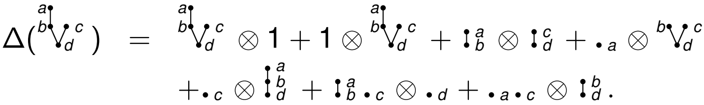
$$X = \sum_{n=1}^{\infty} B_n((1+X)^{n+1}),$$

where B_n is the insertion operator into a primitive Feynman graph with n loops.

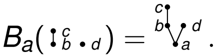
Let I be a set. Set of rooted trees decorated by I:



The Connes-Kreimer construction is extended to obtain the Hopf algebra H_R^I.



1. We assume that I is graded, that is to say there is map $deg : I \longrightarrow \mathbb{N}^*$. Then H_C^I is a graded Hopf algebra, the degree of a forest being the sum of the degree of its decorations.
2. For all $d \in I$, there is a grafting operator $B_d : H_R^I \longrightarrow H_R^I$. For example, if $a, b, c, d \in I$:



Proposition

For all $a \in I$, $x \in H_R^I$:

$$\Delta \circ B_a(x) = B_a(x) \otimes 1 + (Id \otimes B_a) \circ \Delta(x).$$

If I is graded, then for all $a \in I$, B_a is homogeneous of degree $deg(a)$.

Universal property

Let A be a commutative Hopf algebra and for all $a \in I$, let $L_a : A \longrightarrow A$ such that for all $x \in A$:

$$\Delta \circ L_a(x) = L_a(x) \otimes 1 + (Id \otimes L_a) \circ \Delta(x).$$

Then there exists a unique Hopf algebra morphism $\phi : H_R^I \longrightarrow A$ with $\phi \circ B_a = L_a \circ \phi$ for all $a \in A$.

Moreover, if A is graded and if for all $a \in I$, L_a is homogeneous of degree $deg(a)$, then ϕ is homogeneous of degree 0. This allows to lift Dyson-Schwinger equations on Feynman graphs as combinatorial Dyson-Schwinger equations on decorated rooted trees.

Definitions

Let I be a graded set and let $f_i(h) \in K[[h]]$ for all $i \in I$.

- The combinatorial Dyson-Schwinger equations associated to $(f_i(h))_{i \in I}$ is:
$$X = \sum_{i \in I} B_i(f_i(X)),$$
where X lives in the completion of H_R^I.
- This equation has a unique solution $X = \sum X(n)$.
- The subalgebra of H_R^I generated by the $X(n)$'s is denoted by $H_{(f)}$.
- We shall say that the equation is Hopf if $H_{(f)}$ is a Hopf subalgebra.

Lemma

Let us assume that the equation associated to (f) is Hopf. If $f_i(0) = 0$, then $f_i = 0$.

If $f_i(0) = 0$, then $\bullet_i$ does not appear in X, so does not appear in any element of $H_{(f)}$. Moreover:

$$\Delta(X) = X \otimes 1 + 1 \otimes X + f_i(X) \otimes \bullet_i + \ldots \in H_{(f)} \otimes H_{(f)}.$$

So necessarily, $f_i(X) \otimes \bullet_i = 0$, and $f_i = 0$.

We now assume that $f_i(0) = 1$ for all $i \in I$.

Lemma

Let us assume that the equation associated to (f) is Hopf. If $i, j \in I$ have the same degree, then $f_i = f_j$.

Let $n = deg(i) = deg(j)$. Then $X(n) = \bullet_i + \bullet_j + \ldots$. Consequently, in any element of $H_{(f)}$, $\bullet_i$ and $\bullet_j$ have the same coefficient. Moreover:

$$\Delta(X) = X \otimes 1 + 1 \otimes X + f_i(X) \otimes \bullet_i + f_j(X) \otimes \bullet_j + \ldots \in H_{(f)} \otimes H_{(f)}.$$

Hence, $f_i(X) = f_j(X)$, so $f_i = f_j$.

Grouping 1-cocycles by degrees, we now assume that $I \subseteq \mathbb{N}^*$.

Let us choose $i \in I$. We restrict our solution to i, that is to say we delete any tree with a decoration which is not equal to i. The obtained element X' is solution of:

$$X' = B_i(f_i(X')),$$

and this equation is Hopf. By the study of equations with only one 1-cocycle:

Lemma

For all $i \in I$, there exists $\alpha_i, \beta_i \in K$ such that :

$$f_i = \begin{cases} e^{\alpha_i h} \text{ if } \beta_i = 0, \\ (1 - \alpha_i \beta_i h)^{-1/\beta_i} \text{ if } \beta_i \neq 0. \end{cases}$$

Lemma

Let us write:

$$X = \sum_t a_t t.$$

For all $i \in I$, there exists coefficients $\lambda_n^{(i)}$ such that for any rooted tree t:

$$\lambda_{|t|}^{(i)} a_t = \sum_{t'} n_i(t, t') a_{t'},$$

where $n_i(t, t')$ is the number of leaves of t' decorated by i such that the cut of this leaf gives t.

By the study of equations with a single 1-cocycle:

Lemma

If f_i is not constant, then for all $n \geq 1$, for all $j \in I$:

$$\lambda_{ni}^{(j)} = \alpha_i(1 + (n-1)\beta_i).$$

If f_i and f_j are not constant, computing $\lambda_{nij}^{(j)}$ in two different ways:

$$nj\alpha_i(1+\beta_i) - \alpha_i\beta_i = ni\alpha_j(1+\beta_j) - \alpha_j\beta_j.$$

Lemma

There exists $\lambda, \mu \in K$ such that if f_i is not constant, then $\alpha_i = \lambda i - \mu \neq 0$ and $\beta_i = \frac{\mu}{\lambda i - \mu}$.

Proposition

Let (E) be a Hopf Dyson-Schwinger equation. Then I can be written as $I = I' \sqcup I''$, and there exists $\lambda, \mu \in K$, $\lambda \neq 0$, such that if we put:

$$Q(h) = \begin{cases} (1-\mu h)^{-\frac{\lambda}{\mu}} \text{ if } \mu \neq 0, \\ e^{\lambda h} \text{ if } \mu = 0, \end{cases}$$

then:

$$(E) : X = \sum_{j \in I'} B_j\left((1-\mu X)Q(X)^i\right) + \sum_{j \in I''} B_j(1).$$

Lemma

Let us consider a Dyson-Schwinger equation of the form:

$$X = B_i(1) + B_j(f(X)),$$

with f non constant. If it is Hopf, then there exists a non-zero $\alpha \in K$, such that $f(h) = 1 + \alpha h$ or $f(h) = \left(1 - \alpha \frac{i}{j-i} h\right)^{\frac{i-j}{i}}$.

We define inductively a family of trees by $t_1 = \vdots^{\,i}_{\,j}$ and $t_{n+1} = B_j(\bullet_i \, t_n)$ for all $n \geq 1$.

$$\lambda^{(i)}_{n(i+j)}(1+\beta)^{n-1} = (n-1)(1+2\beta)(1+\beta)^{n-1} + (1+\beta)^n.$$

Let us assume that $\beta \neq -1$. Then:

$$\lambda^{(i)}_{n(i+j)} = (n-1)(1+2\beta) + 1 + \beta = n(1+2\beta) - \beta.$$

Compute $\lambda^{(i)}_{j(i+j)}$ in two different ways:

$$\begin{array}{rcl} \lambda^{(i)}_{j(i+j)} & = & \lambda^{(i)}_{(i+j)j} \\ & = & \alpha(1+\beta)(i+j) - \alpha\beta, \\ & = & \lambda^{(i)}_{j(i+j)} \\ & = & \alpha j(1+2\beta) - \alpha\beta. \end{array}$$

Hence, $(1+\beta)(i+j) = j(1+2\beta)$, so $\beta = \frac{i}{j-i}$. As a conclusion, $\beta = -1$ or $\frac{i}{j-i}$, therefore $f(h) = 1 + \alpha h$ or $\left(1 - \alpha\frac{i}{j-i}h\right)^{\frac{i-j}{i}}$.

Lemma

1. Let us consider a Dyson-Schwinger equation of the form:

$$X = B_i(1) + B_j(f(X)) + B_k(g(X)),$$

with f, g non constant. If it is Hopf, then there exists a non-zero $\alpha \in K$, such that $(f = (1 - \alpha i h)^{-\frac{j}{i}+1}$ and $g = (1 - \alpha i h)^{-\frac{k}{i}+1})$ or $(f = g = 1 + \alpha h)$.

2. Let us consider a Dyson-Schwinger equation of the form:

$$X = B_i(1) + B_j(1) + B_k(f(X)),$$

where f is non constant. Then there exists a non-zero $\alpha \in K$, such that $f = 1 + \alpha h$.

Theorem

One of the following assertions holds:

1. there exists $\lambda, \mu \in K$ such that, if we put:

$$Q(h) = \begin{cases} (1-\mu h)^{-\frac{\lambda}{\mu}} \text{ if } \mu \neq 0, \\ e^{\lambda h} \text{ if } \mu = 0, \end{cases}$$

then:

$$(E) : x = \sum_{i \in I} B_j \left((1-\mu x) Q(x)^i \right).$$

2. There exists $m \geq 0$ and $\alpha \in K - \{0\}$ such that:

$$(E) : x = \sum_{\substack{i \in I \\ m \mid i}} B_i(1+\alpha x) + \sum_{\substack{i \in I \\ m \nmid i}} B_i(1).$$

1. Let I be a set. The primitive elements of $(H_R^I)^*$ inherits a prelie structure. Moreover, it is the free prelie algebra generated by $\bullet_i$, $i \in I$.
2. If $I \subseteq \mathbb{N}^*$, there exists a prelie algebra morphism $\phi_\lambda : Prim((H_R^I)^*) \longrightarrow \mathfrak{g}_{FdB}$, sending $\bullet_i$ to e_i for all i.
3. By duality, we obtain a Hopf algebra morphism from $S(\mathfrak{g}_{FdB})^*$ to H_R^I. Its image is generated by the components of the solutions of the Dyson-Schwinger equations of the first type, with parameters $\frac{-1}{\lambda}$ and $\frac{-1-\lambda}{\lambda}$.

Corollary

For all $\lambda, \mu \in K$, the algebra generated by the components of the solution of the Dyson-Schwinger equation of the first type is a Hopf subalgebra.

Corollary

If $\mu \neq -1$ and $\lambda = 1 + \mu$,

$$\Delta(X) = X \otimes 1 + \sum_{j=1}^{\infty} (1 + \lambda' X)^{1+\frac{j}{\lambda'}} \otimes X(j),$$

with $\lambda' = \dfrac{-1}{1+\mu}$.

Description of the prelie algebra in the second case: to simplify, we assume that $1 \in I$.

Theorem

$$X = \sum_{\substack{i \in I \\ m \mid j}} B_i(1 + \alpha X) + \sum_{\substack{i \in I \\ m \nmid i}} B_i(1),$$

with $\alpha \in K - \{0\}$. The dual of $H_{(f)}$ is the enveloping algebra of a pre-Lie algebra $\mathfrak{g}$, such that:

- $\mathfrak{g}$ has a basis $(f_i)_{i \geq 1}$.
- For all $i, j \geq 1$:

$$f_i \circ f_j = \begin{cases} 0 \text{ if } m \nmid j, \\ f_{i+j} \text{ if } m \mid j. \end{cases}$$

The product $\circ$ is associative.