Renormalized products of distributions which fail to satisfy the Hörmander condition

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Plan.

1. Motivation: renormalization of QFT on curved space times.

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2. Renormalization= problem of extension of distributions.
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4. The wave front set: ideas and applications.
5. Taming the wave front set of extensions.

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- Space time M.
- "Correlation functions" of QFT: these are products of distributions on M^n minus the diagonal $d_n = \{\underbrace{(x, \cdots, x)}_{n-\text{times}} | \, x \in M\}$.

Example

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$\langle 0 | T\phi^2(x)\phi^2(y) | 0 \rangle$: *product of distributions, ill defined on* $x = y$ *!*

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- How to make sense of these products **on whole** M^n ?

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Extension of distributions.

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Question: can t be extended to all M ?

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Counterexample showed to me by L. Boutet de Monvel.

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Problem: Grows to fast when $h \to 0$. Need to impose some extra conditions.

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For a distribution t:

$$\forall \varphi \in \mathcal{D}(\mathbb{R}^{n+d}), \langle t_\lambda, \varphi \rangle = \lambda^{-d} \langle t, \varphi_{\lambda^{-1}} \rangle .$$

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Denote this space by $E_s(\mathbb{R}^{n+d} \setminus I)$.

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Denote this space by $E_s(\mathbb{R}^{n+d} \setminus I)$.
Note that the Besov spaces $B^{\gamma}_{\infty,\infty}(\mathbb{R}^d)$ appearing in lots of talks are in $E_s(\mathbb{R}^d)$ for some s.

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$e^{\log \lambda \rho} : (x, h) \mapsto (x, \lambda h)$

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$\rho = h^j \frac{\partial}{\partial h^j}$: scaling transversally to I. Integrate this vector field $e^{t\rho} : (x, h) \mapsto (x, e^t h)$, parametrise with logarithmic time $e^{\log \lambda\rho} : (x, h) \mapsto (x, \lambda h)$ relate scaling of functions with **pull-back** or composition with the above diffeos:

$$\varphi_\lambda = e^{\log \lambda\rho\star}\varphi = \varphi \circ e^{\log \lambda\rho}.$$

The cut–off function χ.

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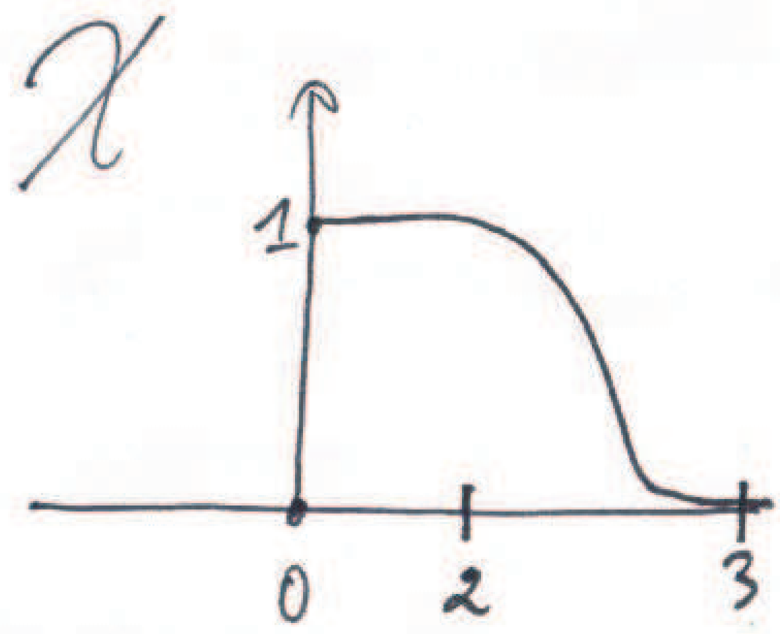

Figure: Graph of χ in h coordinates.

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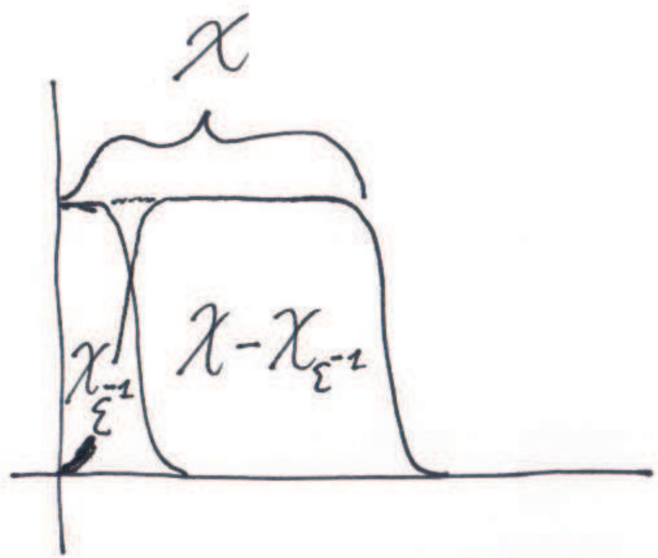

Figure: $\chi - \chi_{\varepsilon^{-1}}$.

The main idea.

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Idea: Does

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$$t(1 - \chi_{\varepsilon^{-1}})$$

have a limit when $\varepsilon \to 0$?

Nice case.

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Theorem

Let $t \in E_s\left(\mathbb{R}^{n+d} \setminus I\right)$, *if* $s+d>0$ *then*

$$\forall \varphi \in \mathcal{D}(\mathbb{R}^{n+d}), \overline{t}(\varphi) = \lim_{\varepsilon \to 0} \langle t(1-\chi_{\varepsilon^{-1}}), \varphi \rangle \tag{1}$$

exists and defines an extension $\overline{t} \in \mathcal{D}'(\mathbb{R}^{n+d})$ *and* $\overline{t}$ *is in* $E_s(\mathbb{R}^{n+d})$.

Example

$d = 1$ and we work in $\mathbb{R}$ where $I = \{h = 0\}$. $t(h) = |h|^{-\frac{1}{2}}$ is L^1_{loc} near $h = 0$ thus for all test function φ,

$\lim_{\varepsilon\to 0} \int_{\mathbb{R}} (1 - \chi_{\varepsilon^{-1}})|h|^{-\frac{1}{2}}\varphi(h)dh = \int_{\mathbb{R}} |h|^{-\frac{1}{2}}\varphi(h)dh.$

Nasty divergent case.

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Theorem

Let $t \in E_s\left(\mathbb{R}^{n+d} \setminus I\right)$, *if* $s + d \in (-m-1, -m], m \in \mathbb{N}$ *then there exists a family* $(c_\varepsilon)_\varepsilon$ *of distributions supported on* I *(called* **local counterterms***) s.t.*

$$\overline{t} = \lim_{\varepsilon \to 0} \left(\left\langle t\left(\chi - \chi_{\varepsilon^{-1}}\right), \varphi\right\rangle - \langle c_\varepsilon, \varphi\rangle\right) + \langle t(1-\chi), \varphi\rangle \qquad (2)$$

exists and defines an extension $\overline{t} \in \mathcal{D}'(\mathbb{R}^{n+d})$. *If* s *is not an integer then the extension* $\overline{t}$ *is in* $E_s(\mathbb{R}^{n+d})$, *otherwise* $\overline{t} \in E_{s'}(\mathbb{R}^{n+d}), \forall s' < s$.

example in the Hadamard style.

Example

$d = 1$ and we work in $\mathbb{R}$ where $I = \{h = 0\}$.
$\lim_{\varepsilon\to 0}\int_\varepsilon^\infty dh\frac{1}{|h|}\varphi(h)$ *does not exist but if we subtract the local counterterm* $\log\varepsilon\delta_0$ *then*

$$\int_\varepsilon^\infty dh\frac{1}{|h|}\varphi(h) - \log\varepsilon\delta_0(\varphi)$$

$$= \int_\varepsilon^1 dh\frac{1}{|h|}(\varphi(h) - \varphi(0)) + \int_1^\infty dh\frac{1}{|h|}\varphi(h)$$

has a limit when $\varepsilon \to 0$*!*

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Definition

A vector field ρ is called Euler if

$$\forall f \in \mathcal{I}, \rho f - f \in \mathcal{I}^2. \tag{3}$$

Nice properties

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Example

$\rho = h^j \frac{\partial}{\partial h^j}$ *is Euler.*

- All Euler vector fields are locally conjugate, i.e. for each $p \in I$, we can always change the coordinates in a neighborhood of p so that $\rho = h^j \frac{\partial}{\partial h^j}$.

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- Distribution t weakly homogeneous at $p \in I$: independent of choice of ρ !
- Can give intrinsic definition of t weakly homogeneous at I.

Theorem

Let U be an open neighborhood of $I \subset M$, if $t \in E_s(U \setminus I)$ then there exists an extension $\overline{t}$ in $E_{s'}(U)$ where $s' = s$ if $-s-d \notin \mathbb{N}$ and $s' < s$ otherwise.

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- *The wave front set of t is a subset* $WF(t) \subset T^\star M \setminus \{\underline{0}\}$.
- *Singular support of* $t = \pi(WF(t))$ *where* $\pi : T^\star M \mapsto M$ *is the canonical projection.*

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- $WF(t) = \emptyset$, *iff* t *is smooth.*

examples and applications.

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$WF(\delta_0)$ is $T_0^\star M \setminus \{0\}$. In $\mathbb{R}^{n+d}$, $I = \{h = 0\} = \mathbb{R}^n \times \{0\}$, define delta distribution δ_I and $WF(\delta_I)$ is the conormal of I.

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WF theory explains why the product $\frac{1}{(x+i0)}\frac{1}{(x+i0)}$ makes sense in $\mathcal{D}'$ and not $\frac{1}{(x+i0)}\frac{1}{(x-i0)}$.
In QFT, we can make sense of the complex powers $\left((x_0+i0)^2 - \sum_{i=1}^3 x_i^2\right)^s$.

Conormal of I: C.

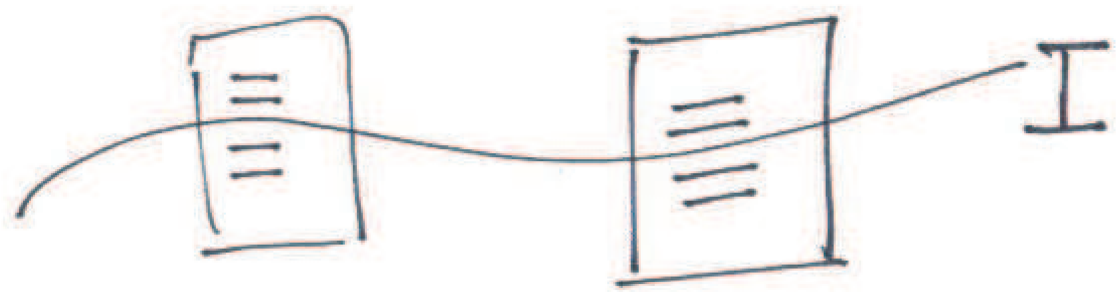

Figure: The conormal bundle of I.

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Theorem

If the extension $\overline{t}$ is in E_s, $-m-1 < s+d \leqslant -m, m \in \mathbb{N}$ and $WF(\overline{t})|_I \subset C$, then the space of extension is a finitely generated module over $C^\infty(I)$ of rank $\frac{m+d!}{m!d!}$.

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- In $T^\star M$ use the notation $(x;\xi)$ where x space and ξ momentum.
- For any two cones Γ_1, Γ_2 in cotangent space, the conic set $\Gamma_1 + \Gamma_2$ denotes the set of all $(x;\xi) = (x;\xi_1 + \xi_2)$, for $(x;\xi_1) \in \Gamma_1, (x;\xi_2) \in \Gamma_2$.

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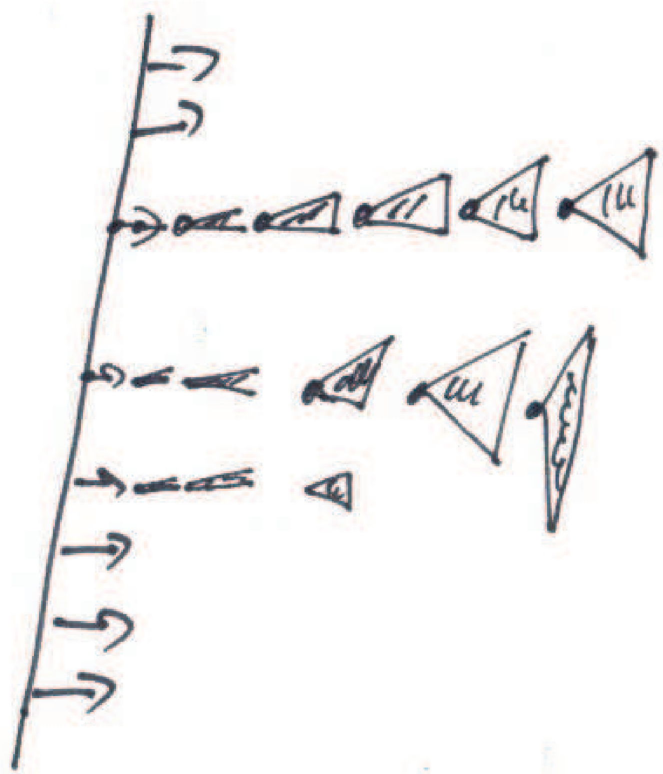

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Let Γ_1, Γ_2 *be two cones in* $T^\bullet(M \setminus I)$. $(\Gamma_i)_{i=1,2}$ *both satisfy the conormal landing condition and* $\Gamma_1 \cap -\Gamma_2 = \emptyset$.

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Let Γ_1, Γ_2 *be two cones in* $T^\bullet(M \setminus I)$. $(\Gamma_i)_{i=1,2}$ *both satisfy the conormal landing condition and* $\Gamma_1 \cap -\Gamma_2 = \emptyset$. *Set* $\Gamma = \Gamma_1 + \Gamma_2 \cup \Gamma_1 \cup \Gamma_2$. *Then for all distributions* $(u_i)_{i=1,2}$ *s.t.* $WF(u_i) \subset \Gamma_i$ *and* $(\lambda^{-s_i} u_{i\lambda})_{\lambda \in (0,1]}$ *is bounded in* $D'_{\Gamma_i}(M \setminus I)$ *for some* $s_i \in \mathbb{R}$, *then*

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and $\lambda^{-s}\overline{u_1 u_2}_\lambda$ *is bounded in* $D'_{\Gamma \cup C}(M)$ *for all* $s < s_1 + s_2$.

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- Thanks for your attention.