S-MATRIX AND RENORMALIZATION

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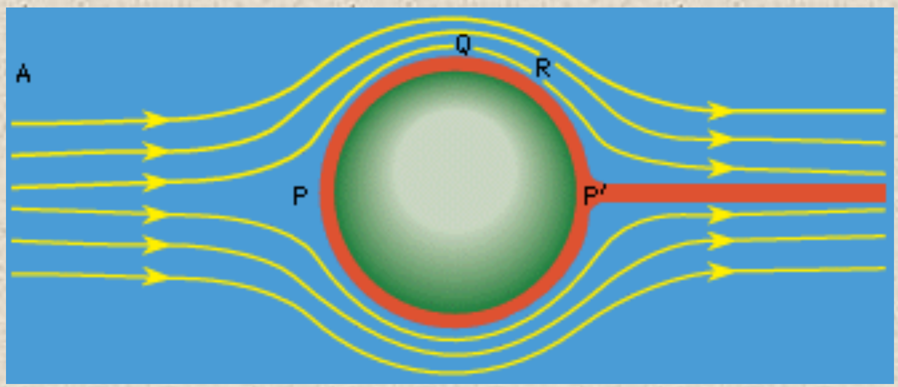
OUTLINE

- Renormalization describes a change of the interaction
- Renormalization removes infinities
- The S-matrix and causality
- Causality, renormalization and changing the interaction

MASS RENORMALIZATION

- Dynamics of a sphere in a fluid



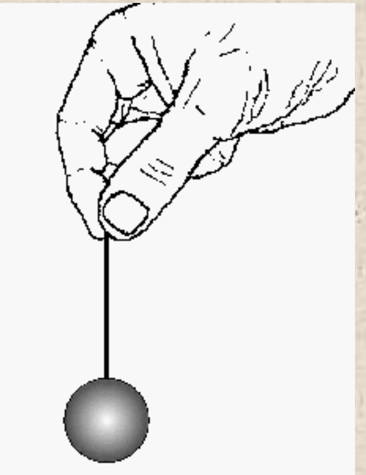
- Chevalier Pierre du Buat (1786)
- Friedrich Wilhelm Bessel (1829)
- Siméon Denis Poisson (1832)
- Giovanni Plana (1835)
- George Green (1835)
- George Gabriel Stokes (1851)

- The equation of motion of a sphere in a fluid submitted to a force $\mathbf{F}$ is

$$\mathbf{F} = m\ddot{\mathbf{x}}$$

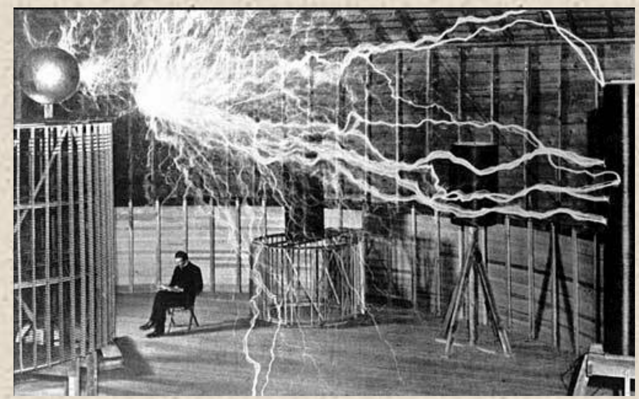
where

$$m = m_0 + \frac{1}{2} V \rho_{\text{fluid}}$$



Charge renormalization

- Apply an external current $\mathbf{j}_{\text{ext}}$



- Assume that the response of the system is

$$\delta^{(0)}\mathbf{j} = \delta\mathbf{j} + \alpha\mathbf{j}_{\text{ext}}$$

- The total current is

$$\mathbf{j}_T = \delta^{(0)}\mathbf{j} + \mathbf{j}_{\text{ext}} = \delta\mathbf{j} + (1+\alpha)\mathbf{j}_{\text{ext}}$$

- The charge has been renormalized by $1+\alpha$

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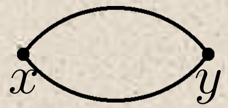
Removing Infinities

Feynman propagator $p^2 = (p^0)^2 - (p^1)^2 - (p^3)^2 - (p^4)^2$

$$D_F(x,y) = i\langle 0|T\big(\varphi(x),\varphi(y)\big)|0\rangle = -\int_{\mathbb{R}^4} \frac{dp}{(2\pi)^4}\frac{e^{-ip\cdot(x-y)}}{p^2 - m^2 + i\epsilon}$$

Feynman diagram

- Example: $\langle 0|T\big(\varphi^2(x),\varphi^2(y)\big)|0\rangle = -2D_F^2(x,y)$

$D_F^2(x,y) \quad =$ 

$$D_F^2(x,y) = \int_{\mathbb{R}^4\times\mathbb{R}^4} \frac{dp}{(2\pi)^4}\frac{dk}{(2\pi)^4}\frac{e^{-ip\cdot(x-y)}}{(k^2 - m^2 + i\epsilon)((p-k)^2 - m^2 + i\epsilon)}$$

- The integral is divergent

Removing Infinities

We rewrite
$$D_F^2(x,y) = \int_{\mathbb{R}^4\times\mathbb{R}^4} \frac{dp}{(2\pi)^4}\frac{dk}{(2\pi)^4} e^{-ip\cdot(x-y)} F(p,k)$$

where
$$F(p,k) = \frac{1}{(k^2-m^2+i\epsilon)((p-k)^2-m^2+i\epsilon)}$$

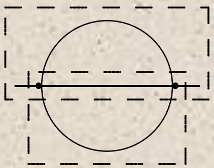
- The degree of the denominator is 4: logarithmic divergence
- Replace $F(p,k)$ by the convergent

$$F(p,k) - F(0,k) = \frac{2p\cdot k - p^2}{(k^2-m^2+i\epsilon)^2((p-k)^2-m^2+i\epsilon)}$$

- The counter-term is local $\delta(x-y)(2\pi)^{-4}\int dk F(0,k)$

HISTORY

- Dyson (1949): problem of overlapping divergences



$$D_F^3(x-y) = \int \frac{(2\pi)^{-12} dk_1 dk_2 dp\ e^{ip.(x-y)}}{(k_1^2 - m^2 + i\epsilon)(k_2^2 - m^2 + i\epsilon)((p - k_1 - k_2)^2 - m^2 + i\epsilon)}$$

- Salam (1951): solution of this problem
- Bogoliubov and Shirkov (1955) correct solution
- Bogoliubov and Parasiuk (1957) all-order proof
- Hepp (1966) correction of the proof
- Standard BPH(Z) renormalization

HISTORY

« *Renormalization theory has a history of egregious errors by distinguished savants. It has a justified reputation of perversity; a method that works up to 13th order in the perturbation series fails in the 14th order.* »

A. S. Wightman Renormalization Theory (1976)

« *As one of the inventors, I remember that we thought of QED in1949 as a temporary and jerry-built structure, with mathematical inconsistencies and renormalized infinities swept under the rug. We did not expect it to last more than ten years before some more solidly built theory would replace it.* »

Freeman Dyson, Letter to Gerald Gabrielse 2006

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S-Matrix and Causality

- Problem: formulate perturbative quantum field theory using only well-defined objects
- The causal approach
- Contributions of
  - Heisenberg: the S-matrix (1943)
  - Stueckelberg and Rivier: causality and extension (1950)
  - Bogoliubov: extension (1952)
  - Bogoliubov and Shirkov: causality and extension (1955)
  - Epstein and Glaser: mathematically correct (1973)
  - Brunetti and Fredenhagen: curved spacetime (2000)

THE S-MATRIX

- Wheeler (1937), Heisenberg (1943)
- Evolution operator $$i\frac{\partial U(t;H)}{\partial t} = H(t) \star U(t;H)$$
- Boundary condition $U(-\infty;H) = \mathrm{Id}$
- Definition $S(H) = U(+\infty;H)$
- Physical meaning: (finite) scattering amplitude
- Causality (Stueckelberg): if $\mathrm{supp}(H_1) > \mathrm{supp}(H_2)$ then

$$U(t,H_1+H_2) = U(t,H_1) \star U(t,H_2)$$

$$S(H_1+H_2) = S(H_1) \star S(H_2)$$

Star-product

- For $A = \varphi^{n_1}(x_1)\ldots\varphi^{n_p}(x_p)$, $B = \varphi^{m_1}(y_1)\ldots\varphi^{m_q}(y_q)$

  the product is (Dütsch and Fredenhagen)

$$(A \star B)(\varphi) = e^{\int dxdy D_+(x,y) \frac{\delta^2}{\delta\varphi_1(x)\delta\varphi_2(y)}} A(\varphi_1)B(\varphi_2)|_{\varphi_1=\varphi_2=\varphi}$$

- Hopf algebraic interpretation (Borcherds)

$$A \star B = \sum (A_{(1)}|B_{(1)})A_{(2)}B_{(2)}$$

- The Laplace pairing: $(\varphi|\varphi) = D_+$

$$(AB|C) = \sum (A|C_{(1)})(B|C_{(2)}) \quad (A|BC) = \sum (A_{(1)}|B)(A_{(2)}|C)$$

- Wick's theorem

Time-Ordering Operator

- Picard iteration

$$S(H) = 1 - i\int_{-\infty}^{\infty} H(t_1)dt_1 - \int_{-\infty}^{\infty}\int_{-\infty}^{t_1} H(t_1)\star H(t_2)dt_1dt_2 + \ldots$$

- Time ordering (Dyson iteration)

$$S(H) = 1 + \sum_{n=1}^{\infty}\frac{(-i)^n}{n!}\int_{-\infty}^{+\infty}\ldots\int_{-\infty}^{+\infty} T\big(H(t_1)\ldots H(t_n)\big)dt_1\ldots dt_n$$

- Definition: $T\big(H(t_1)\ldots H(t_n)\big) = H(t_{\sigma(1)})\star\cdots\star H(t_{\sigma(n)})$

  when $t_{\sigma(1)} > \cdots > t_{\sigma(n)}$

- Renormalization: what happens when $t_i = t_j$
- Consequence of the causality relation:

$$T(AB) = T(A)\star T(B)$$

when $t_i > t'_j$

$$A = \mathcal{H}_1(t_1)\ldots\mathcal{H}_k(t_k)$$
$$B = \mathcal{H}_{k+1}(t'_1)\ldots\mathcal{H}_{k+n}(t'_n)$$

Relativistic Causality

- Hamiltonian density $\mathcal{H}(x)$ where $x = (t, \mathbf{r})$

$$H(t) = \int_{\Sigma(t)} \mathcal{H}(x) d\mathbf{r}$$

- Algebraic locality: if x and y are space-separated points, then $\mathcal{H}(x) \star \mathcal{H}(y) = \mathcal{H}(y) \star \mathcal{H}(x)$
- Algebraic causality (Bogoliubov): Let

$$A = \mathcal{H}_1(x_1) \ldots \mathcal{H}_k(x_k) \qquad B = \mathcal{H}_{k+1}(y_1) \ldots \mathcal{H}_{k+n}(y_n)$$

then

$$T(AB) = T(A) \star T(B)$$

when no x_i is in the causal past of any y_j

Simple Example

- Causality: $x = (t, \mathbf{r})$

$$T\big(\varphi(x)\varphi(0)\big) = \varphi(x)\star\varphi(0) \text{ if } t>0$$

$$T\big(\varphi(x)\varphi(0)\big) = \varphi(0)\star\varphi(x) \text{ if } t<0$$

- Feynman's propagator: $D_F(x) = \langle 0|T\big(\varphi(x)\varphi(0)\big)|0\rangle$
- Causality implies

$$D_F(x) = i\langle 0|\varphi(x)\star\varphi(0)|0\rangle = D_+(x) \quad \text{if} \quad t>0$$

$$D_F(x) = i\langle 0|\varphi(0)\star\varphi(x)|0\rangle = D_-(x) = D_+(-x) \quad \text{if} \quad t<0$$

- Feynman in terms of Wightman propagators
- Massless case $D_\pm(x) = \dfrac{1}{4\pi^2}\dfrac{1}{(t\mp i\epsilon)^2 - r^2}$

Simple Example

- Equivalent form:

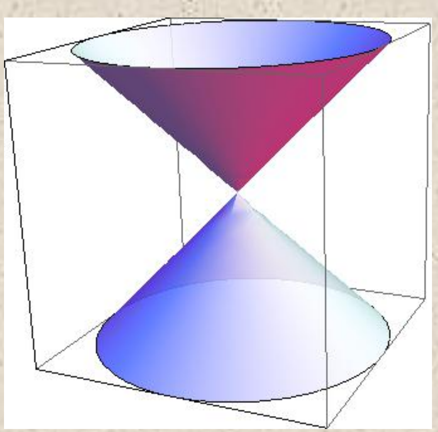
$$D_{\pm}(x) = P\frac{1}{4\pi^2 x^2} \pm \frac{i}{4\pi}\text{sign}(t)\delta(x^2)$$

- Three open sets

$$D_F = D_+ \quad \text{on} \quad \{x; x^2 \geq 0, t > 0\}$$

$$D_F = D_- \quad \text{on} \quad \{x; x^2 \geq 0, t < 0\}$$

$$D_F = D_+ = D_- \quad \text{on} \quad \{x; x^2 < 0\}$$



- The union of the three sets is $\mathbb{R}^4 \backslash \{0\}$

$$D_F = D_+ \quad \text{on} \quad \{x; x^2 < 0 \text{ or } x^2 \geq 0 \text{ and } t > 0\}$$

$$D_F = D_- \quad \text{on} \quad \{x; x^2 < 0 \text{ or } x^2 \geq 0 \text{ and } t < 0\}$$

Simple Example

- How to calculate D_F^n?
- Define D_F^n on $\mathbb{R}^4\backslash\{0\}$ by
  - If x is not in the past of 0:

    $$D_F^n = D_+^n \quad \text{on} \quad \{x; t > 0 \text{ or } x^2 < 0\}$$

  - If 0 is not in the past of x

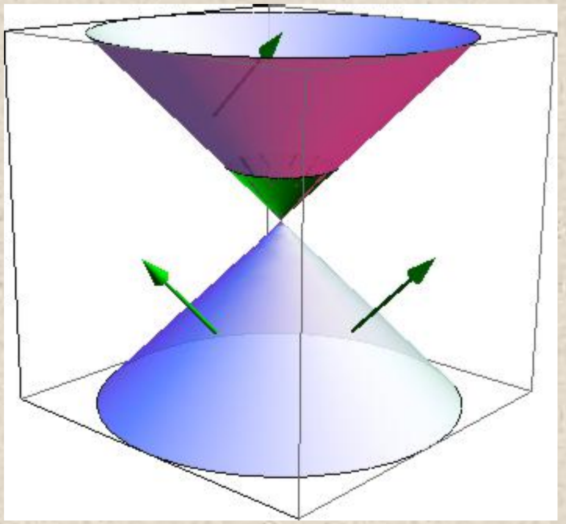
    $$D_F^n = D_-^n \quad \text{on} \quad \{x; t < 0 \text{ or } x^2 < 0\}$$

- The powers $D_\pm^n$ are well defined because the wavefront set satisfies Hörmander's condition
- D_F^n is extended from $\mathcal{D}(\mathbb{R}^4\backslash\{0\})$ to $\mathcal{D}(\mathbb{R}^4)$

Wave-front set

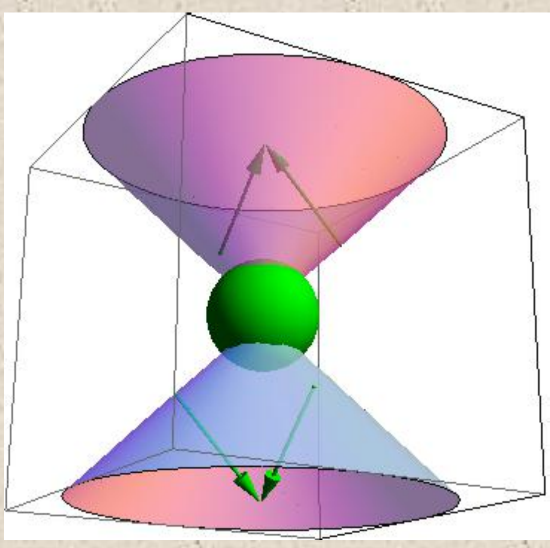
- Wightman propagator

$$\mathrm{WF}(D_+)$$



- Feynman propagator

$$\mathrm{WF}(D_F)$$



Extension Ambiguity

- Causality determines D_F^n on $\mathbb{R}^4 \backslash \{0\}$
- Then, D_F^n is extended from $\mathcal{D}(\mathbb{R}^4 \backslash \{0\})$ to $\mathcal{D}(\mathbb{R}^4)$
- This extension is not unique
- Two extensions differ by a distribution supported on the origin

$$(D_F^n)' = D_F^n + P(\partial)\delta_0$$

- If the degree of divergence of the extension is minimum, then the degree of P is $2n - 4$

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Geometric Lemma

- Popineau and Stora (1982), Bergbauer (2004)
- Let $(S, \leqslant)$ be a poset (a set with a partial order)
- Let I be a proper subset of $\{1, \ldots, n\}$
- Let $C_I = \{(x_1, \ldots, x_n) \in S^n; x_i \not\leqslant x_j \text{ for } i \in I, j \in I^c\}$
- Then,

$$\bigcup_I C_I = S^n \backslash D_n$$

where $D_n = \{(x_1, \ldots, x_n) \in S^n; x_1 = \cdots = x_n\}$

RENORMALIZATION

- For every n, build a linear map

  $$T : C^\infty(M^n) \otimes \mathbb{R}[\varphi_1, \ldots, \varphi_n] \to \mathcal{D}'(M^n) \otimes \mathbb{R}[\varphi_1, \ldots, \varphi_n]$$

- T is uniquely determined if all x_i are different
- Induction on the number of points n
- Start with $T(1) = 1$ and $T(\varphi^k) = \varphi^k$
- On each $C_I \subset M^n$ causality implies

  $$T\big(\prod_{i=1}^n \mathcal{H}_i(x_i)\big) = T\big(\prod_{i\in I} \mathcal{H}_i(x_i)\big) \star T\big(\prod_{j\in I^c} \mathcal{H}_j(x_j)\big)$$

- By Stora's lemma, this determines T on $M^n \backslash D_n$
- Extend T to M^n

Determination of Time-Orderings

- Problem: determine all ways to extend the time-ordering to coinciding points
- A renormalization group acts freely and transitively on the time orderings
- Contributions of
  - Stueckelberg and Peterman (1953)
  - Bogoliubov and Shirkov (1955)
  - Popineau and Stora (1982)
  - Hollands and Wald (2003)
  - Brunetti and Fredenhagen (2009)
  - Borcherds (2011)

TWO POINTS

- A time-ordering on M^2

$$T : C^\infty(M^2) \otimes \mathbb{R}[\varphi_1, \varphi_2] \to \mathcal{D}'(M^2) \otimes \mathbb{R}[\varphi_1, \varphi_2]$$

- On $C_{\{1\}} = \{(x_1, x_2); x_1 \text{ is not in the past of } x_2\}$

$$T(\varphi_1^n \varphi_2^m) = T(\varphi_1^n) \star T(\varphi_2^m) = \varphi_1^n \star \varphi_2^m$$
$$= \sum_i \binom{n}{i} \binom{m}{i} i! D_{12}^i \varphi_1^{n-i} \varphi_2^{m-i}$$

- On $C_{\{2\}} = \{(x_1, x_2); x_2 \text{ is not in the past of } x_1\}$

$$T(\varphi_1^n \varphi_2^m) = \varphi_2^m \star \varphi_1^n = \sum_i \binom{n}{i} \binom{m}{i} i! D_{21}^i \varphi_1^{n-i} \varphi_2^{m-i}$$

where

$$D_{12}(x_1, x_2) = D_+(x_1, x_2) \qquad D_{21}(x_1, x_2) = D_+(x_2, x_1)$$

TWO POINTS

- On $C_{\{1\}} \cup C_{\{2\}} = M^2 \backslash D_2$

$$T(\varphi_1^n \varphi_2^n) = \sum_{i=0}^{n} \binom{n}{i}^2 i!\, D_F^i\, \varphi_1^{n-i} \varphi_2^{n-i}$$

where $D_F(x_1, x_2) = D_+(x_1, x_2)$ on $C_{\{1\}}$

and $D_F(x_1, x_2) = D_+(x_2, x_1)$ on $C_{\{2\}}$

- The extension of $T(\varphi_1^n \varphi_2^m)$ boils down to the extension of D_F^i
- This extension must be done consistently

$$\langle 0|T(\varphi_1^n \varphi_2^n)|0\rangle = n! D_F^n$$

TWO POINTS

- Difference between two extensions $T' - T = \Lambda$
- In terms of extensions of D_F^n

$$\langle 0|\Lambda(\varphi_1^n \varphi_2^n)|0\rangle = n!\big((D_F^n)' - D_F^n\big)$$

- Wick expansion for the difference

$$\Lambda(\varphi_1^n \varphi_2^n) = \sum_{i=0}^{n} \binom{n}{i}^2 \langle 0|\Lambda(\varphi_1^i \varphi_2^i)|0\rangle \, \varphi_1^{n-i} \varphi_2^{n-i}$$

- Λ is supported on the diagonal
- Thus, $T \circ \Lambda = T' \circ \Lambda = \Lambda$

THREE POINTS

- Difference between two extensions

$$\begin{aligned} T(ABC) &= T(A)\star T(BC) \quad \text{on} \quad C_1, \\ &= T(B)\star T(AC) \quad \text{on} \quad C_2, \\ &= T(C)\star T(BC) \quad \text{on} \quad C_3, \\ &= T(AB)\star T(C) \quad \text{on} \quad C_{12}, \\ &= T(AC)\star T(B) \quad \text{on} \quad C_{13}, \\ &= T(BC)\star T(A) \quad \text{on} \quad C_{23} \end{aligned} \qquad \begin{aligned} T'(ABC) &= T'(A)\star T'(BC) \\ &= T'(B)\star T'(AC) \\ &= T'(C)\star T'(BC) \\ &= T'(AB)\star T'(C) \\ &= T'(AC)\star T'(B) \\ &= T'(BC)\star T'(A) \end{aligned}$$

- Example: $$\begin{aligned} T'(A)\star T'(BC) &= T(A)\star T(BC) + T(A)\star \Lambda(BC) \\ &= T(ABC) + T(A\Lambda(BC)) \quad \text{on} \quad C_1 \end{aligned}$$
- Result

$$T'(ABC) = T(ABC) + T(A\Lambda(BC)) + T(B\Lambda(AC)) + T(C\Lambda(AB)) + \Lambda(ABC)$$

where $\Lambda(ABC)$ is supported on the diagonal D_3

EXAMPLE

- For the example of the φ^4 theory

$$\Lambda\Big(\varphi^4(x_1)\dots\varphi^4(x_n)\Big) = \int dx\delta(x_1 - x)\dots\delta(x_n - x)$$
$$\Big(C_{n1}\varphi^2(x) + C_{n2}\varphi^4(x) + C_{n3}\varphi(x)\Box\varphi(x)\Big)$$

- Subtle problem due to balanced derivatives (Dütsch and Fredenhagen, 2004)

General Case

- For $A = \mathcal{H}_1(x_1)\ldots\mathcal{H}_n(x_n)$ and $I \subset \{1,\ldots,n\}$ define

$$A|_I = \prod_{i\in I} \mathcal{H}_i(x_i)$$

- Let

$$e^{*\Lambda}(A) = \Lambda(A) + \sum_{k=2}^{\infty} \sum_{I_1\cup\cdots\cup I_k = I} \Lambda(A|_{I_1})\ldots\Lambda(A|_{I_k})$$

- Then

$$T'(A) = T\big(e^{*\Lambda}(A)\big)$$

- All Λ are supported on the diagonal
- Boundary conditions

$$\Lambda(1) = 0 \qquad\qquad \Lambda(\varphi^n) = \varphi^n$$

Ambiguity of the S-matrix

- The S-matrix corresponding to the interaction $\mathcal{H}(x)$ is

$$S = T(e^A) \quad \text{where} \quad A = -i\int \mathcal{H}(x)g(x)dx$$

- For another time-ordering (another renormalization)

$$S' = T'(e^A) = T(e^{\Lambda(e^A)})$$

- A change of renormalization amounts to a modification of the interaction

$$A \to A' = e^{\Lambda(e^A)} = A + \sum_{n=2}^{\infty} \frac{1}{n!}\Lambda(A^n)$$

- Renormalization provides a way to describe a modification of the interaction

Renormalization group

- The transition from one extension to the other is described by the action of Λ : $T' = T \circ e^{*\Lambda}$
- The set of Λ forms a group for the product

$$\Lambda' \bullet \Lambda = \Lambda' \circ e^{*\Lambda}$$

- This renormalization group acts freely and transitively on the time-ordered products
- It contains all the renormalization groups used in physics (Brunetti, Dütsch, Fredenhagen 2009)

CONCLUSION

- Renormalization is naturally a change of the interaction
- Causality (or a poset) is crucial
- Gauge theory (Hollands 2008, Fredenhagen and Rejzner 2013)
- Gravitation (Brunetti, Fredenhagen, Rejzner 2013)
- Open problem: explicit calculation of S-matrix elements (partition of unity)