Measure-theoretic rejection sampling

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Proposition. *Suppose that* μ, ν *and* ρ *are probability measures on a measurable space* $(E, \mathscr{E})$*; that* $\{\mu, \nu\} \ll \rho$*; and that there is some* $M > 0$ *with the property that*

$$\frac{\mathrm{d}\mu}{\mathrm{d}\rho} \leq M \frac{\mathrm{d}\nu}{\mathrm{d}\rho}, \quad \rho\text{-a.s.} \tag{1}$$

Let $(\Omega, \mathscr{F}, \mathbb{P})$ *be a probability space on which* $X \sim \nu$ *and* $U \sim \mathrm{Unif}([0,1])$ *are defined and independent. Then* μ *is the conditional law of* X *given the event*

$$\frac{\mathrm{d}\mu}{\mathrm{d}\rho}(X) \geq MU \frac{\mathrm{d}\nu}{\mathrm{d}\rho}(X). \tag{2}$$

Proof. (Firstly, note that (1) implies that $\mu \ll \nu$, and that the event (2) occurs with positive probability.) Fix a set $A \in \mathscr{E}$ and observe that

$$\mathbb{P}\left(X \in A \;\middle|\; \frac{\mathrm{d}\mu}{\mathrm{d}\rho}(X) \geq MU \frac{\mathrm{d}\nu}{\mathrm{d}\rho}(X) \right) = \frac{\psi(A)}{\psi(E)}$$

where ψ is defined to be

$$\begin{aligned}
\psi(A) &:= \mathbb{P}\left(X^{-1}(A) \cap \left\{ \frac{\mathrm{d}\mu}{\mathrm{d}\rho}(X) \geq MU \frac{\mathrm{d}\nu}{\mathrm{d}\rho}(X) \right\} \right) \\
&= \int_A \mathbb{P}\left(\frac{\mathrm{d}\mu}{\mathrm{d}\rho}(x) \geq MU \frac{\mathrm{d}\nu}{\mathrm{d}\rho}(x) \right) \nu(\mathrm{d}x) \\
&= \int_{A \cap \{\mathrm{d}\nu/\mathrm{d}\rho > 0\}} \mathbb{P}\left(\frac{\mathrm{d}\mu}{\mathrm{d}\rho}(x) \geq MU \frac{\mathrm{d}\nu}{\mathrm{d}\rho}(x) \right) \frac{\mathrm{d}\nu}{\mathrm{d}\rho}(x)\, \rho(\mathrm{d}x) \\
&= \int_{A \cap \{\mathrm{d}\nu/\mathrm{d}\rho > 0\}} \frac{1}{M} \frac{\mathrm{d}\mu}{\mathrm{d}\nu} \frac{\mathrm{d}\nu}{\mathrm{d}\rho}\, \mathrm{d}\rho \\
&= \frac{1}{M} \mu\left(A \cap \left\{ \frac{\mathrm{d}\nu}{\mathrm{d}\rho} > 0 \right\} \right) = \mu(A)/M.
\end{aligned}$$

(For the last equality, note that

$$\mu\left(\frac{\mathrm{d}\nu}{\mathrm{d}\rho} > 0 \right) = \int_{\{\mathrm{d}\nu/\mathrm{d}\rho > 0\}} \frac{\mathrm{d}\mu}{\mathrm{d}\nu} \frac{\mathrm{d}\nu}{\mathrm{d}\rho}\, \mathrm{d}\rho = \int_E \frac{\mathrm{d}\mu}{\mathrm{d}\nu} \frac{\mathrm{d}\nu}{\mathrm{d}\rho}\, \mathrm{d}\rho = \mu(E)$$

which equals 1.) Hence, $\psi(A)/\psi(E) = M\mu(A)/(M\mu(E)) = \mu(A)$, as required. □

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In exactly the same way as in Alexandra's Monte Carlo notes, samples from this conditional measure can be generated via rejection sampling: let $(X_n)_{n\geq 1}$ and $(U_n)_{n\geq 1}$ be independent iid sequences of random variables, with $X_n \sim \nu$ and $U_n \sim \mathrm{Unif}([0,1])$ for each $n \geq 1$, and define

$$T \coloneqq \inf\left\{ n \geq 1 : \frac{\mathrm{d}\mu}{\mathrm{d}\rho}(X_n) \geq M U_n \frac{\mathrm{d}\nu}{\mathrm{d}\rho}(X_n) \right\}.$$

Then T is almost surely finite and $X_T \sim \mu$.